

Outline

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Introduction to seismic interferometry

2

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2-D and 3-D examples

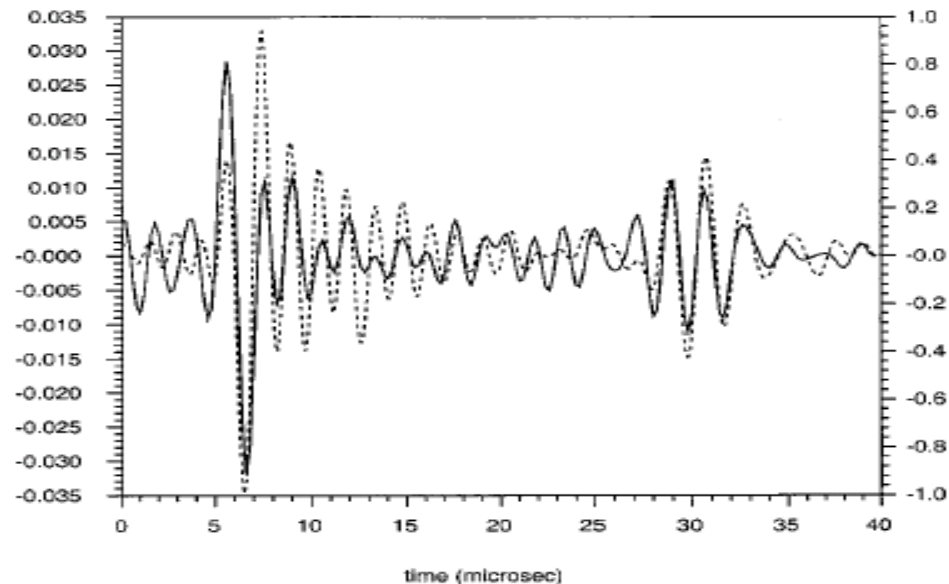
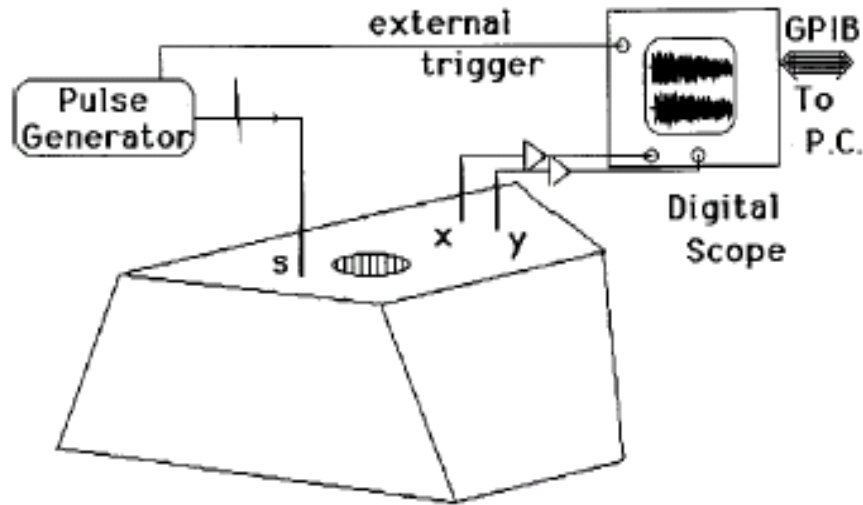
1. Introduction to seismic interferometry

- **Aki (1957, Bull. Earthqu. Res. Inst): SPAC (spatial autocorrelation method)** 圆形台阵噪声信号的空间—时间互相关叠加恢复台阵下面波信号
- **Claerbout (1968, Geophysics):** 通过地表台站接收到的信号的自相关来恢复台站下方层状介质的响应函数（格林函数）
- **Cox (1973, JASA): 2-D / 3-D 噪声源信号（平面波假设）的互相关理论（JASA）**

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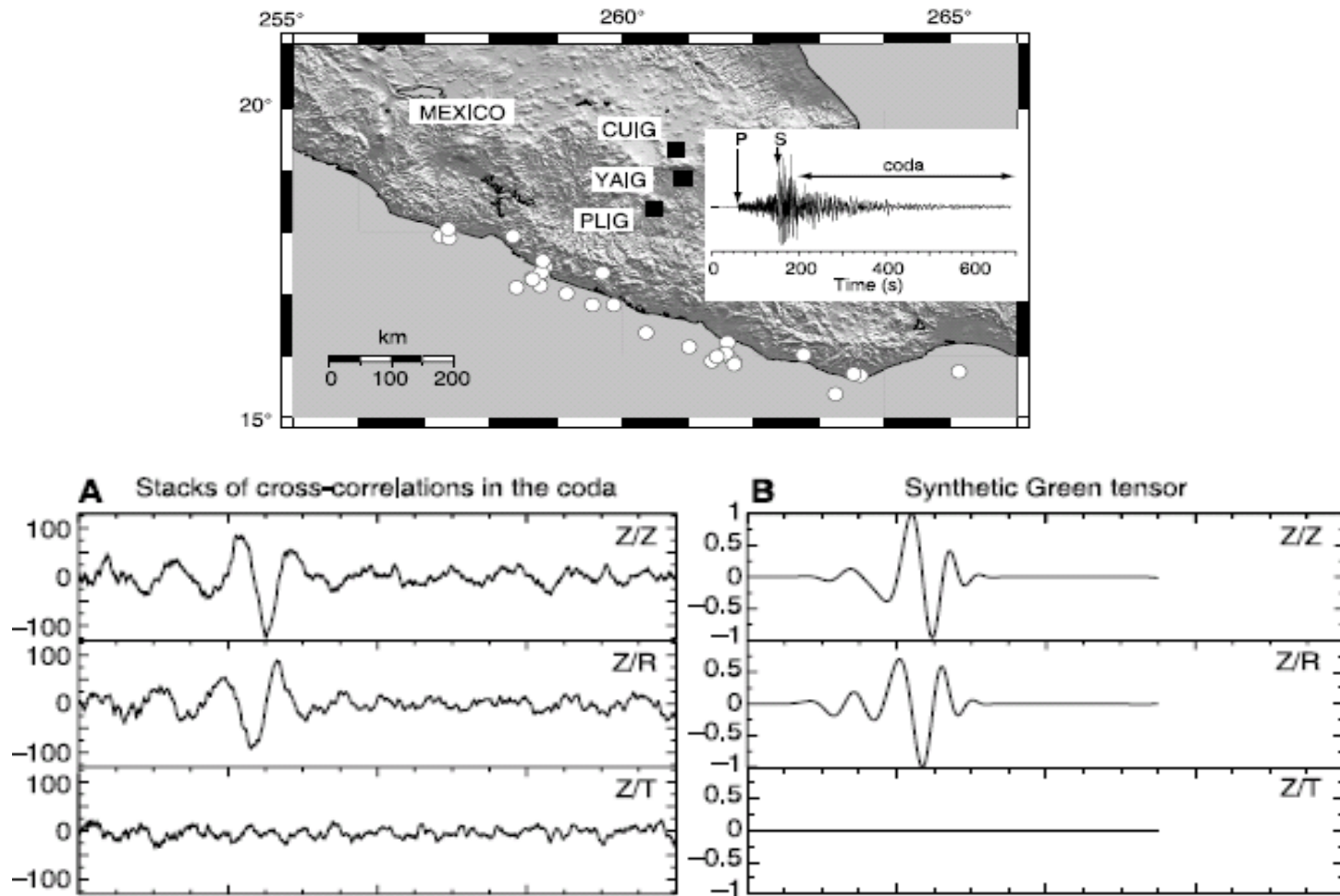
1. Introduction to seismic interferometry

- Ultrasonic experiment by Lobkis & Weaver (2001, JASA)



1. Introduction to seismic interferometry

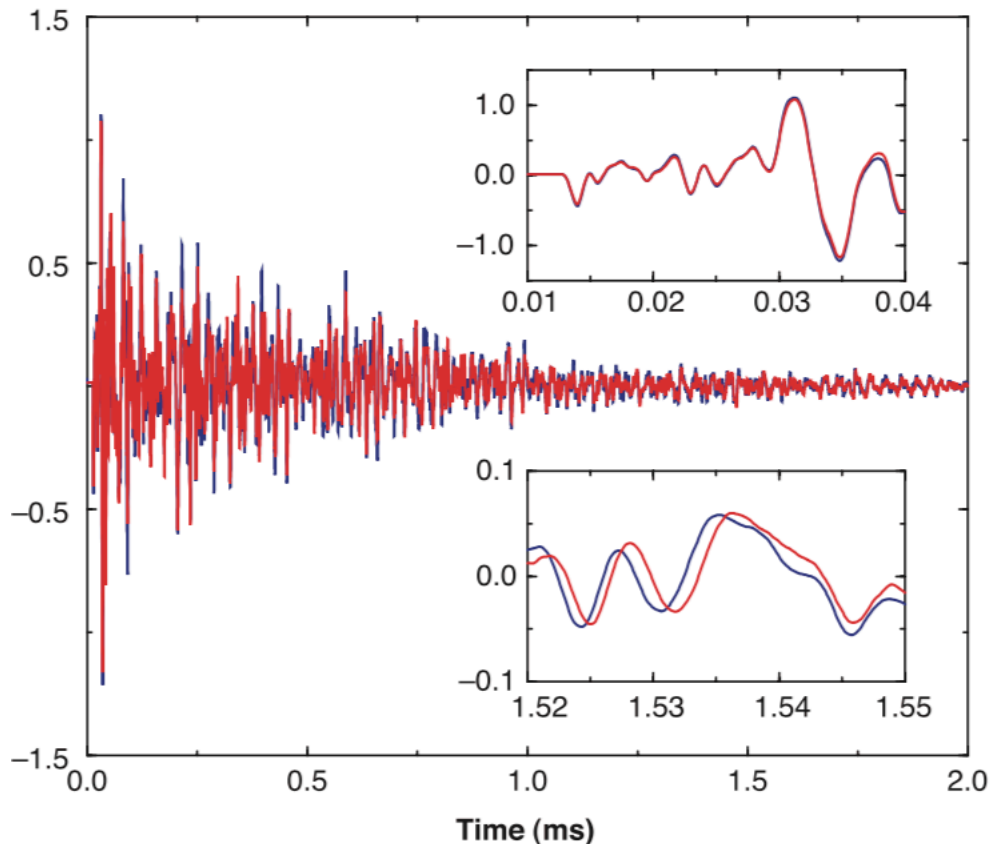
- Coda wave interferometry



(Campillo & Paul, 2003, Science)

1. Introduction to seismic interferometry

- Coda wave interferometry

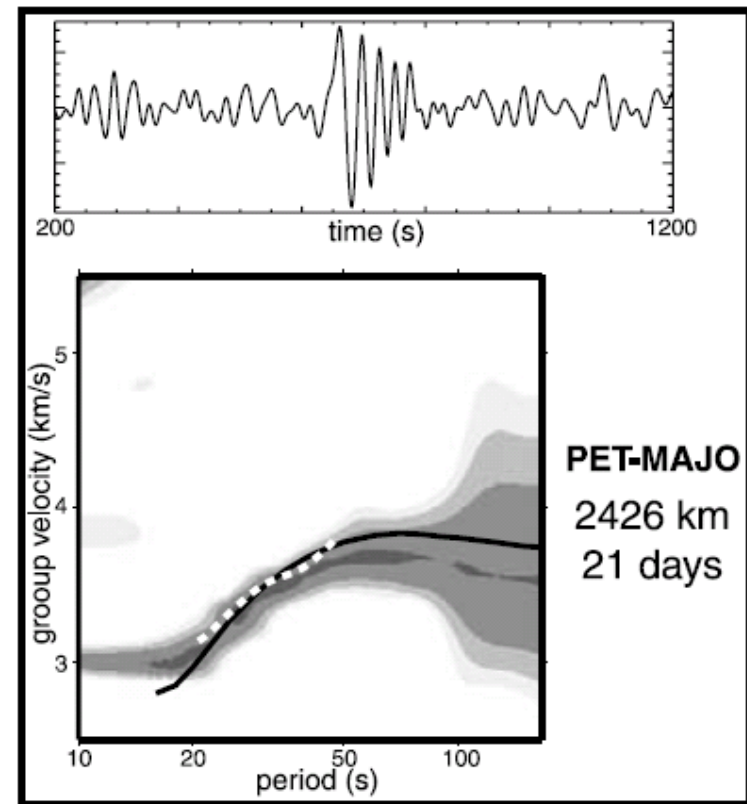
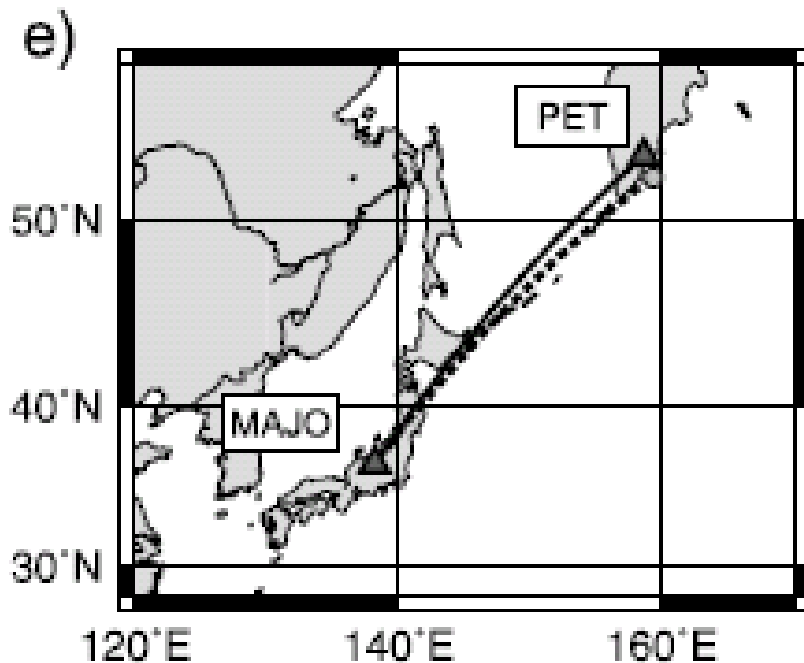


$$\frac{dv}{v} = - \frac{\langle \tau \rangle}{t}$$

(Snieder et al., 2002, Science)

1. Introduction to seismic interferometry

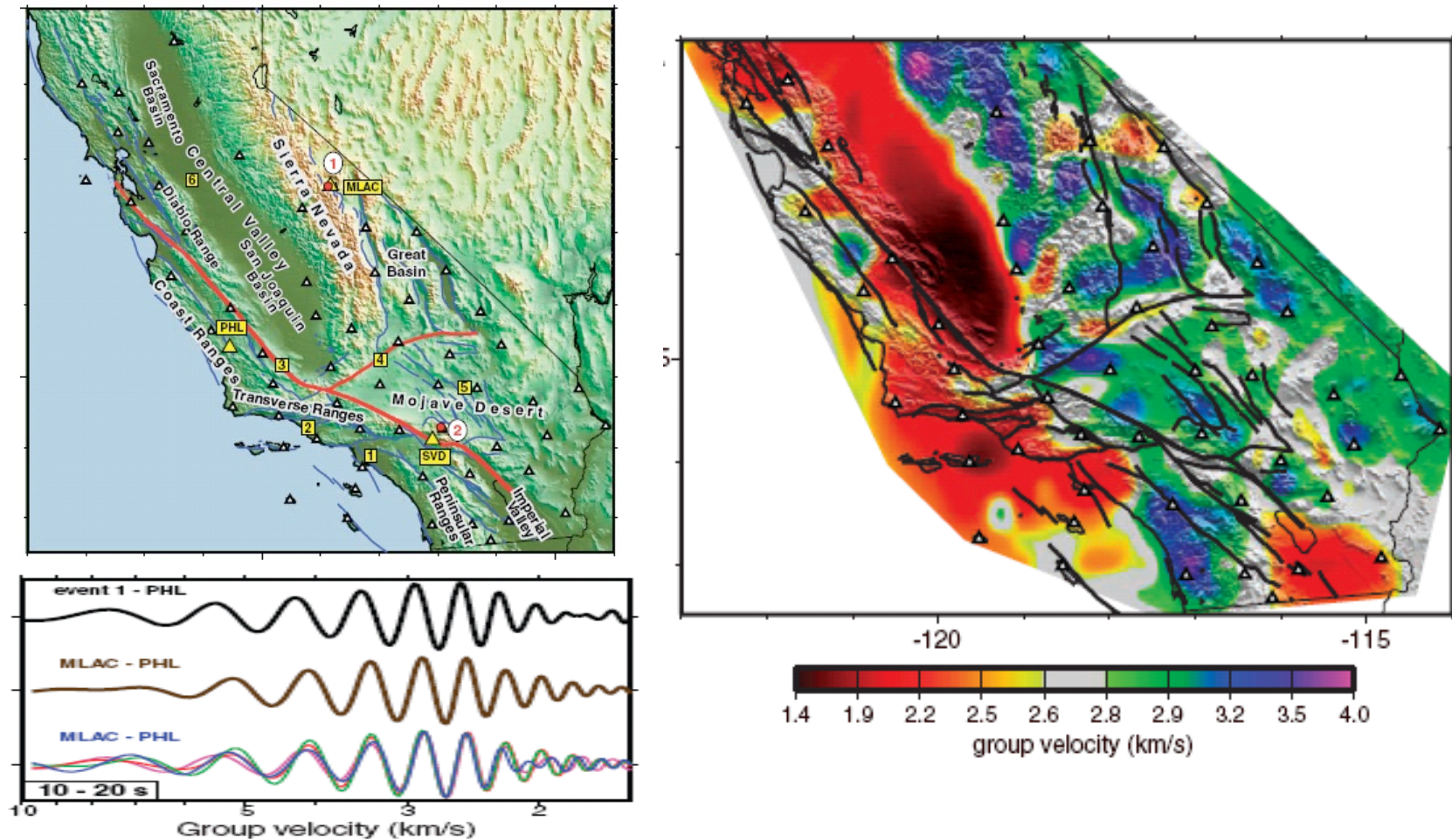
- Ambient noise interferometry



(Shapiro & Campillo, GRL, 2004)

1. Introduction to seismic interferometry

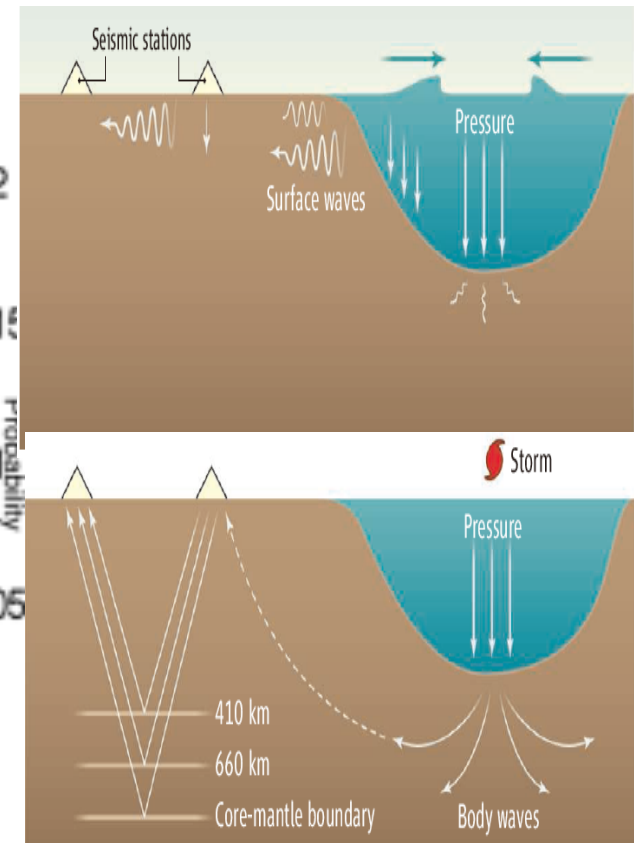
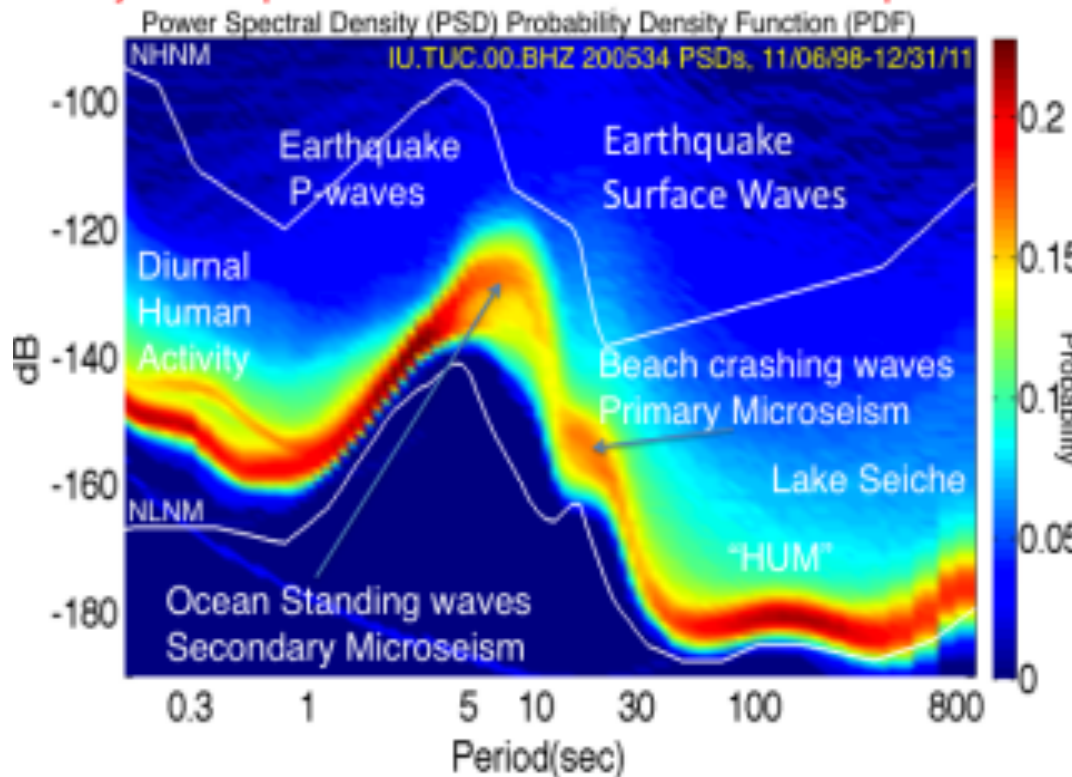
- Ambient noise interferometry



(Shapiro et al., 2005, Science)

1. Introduction to seismic interferometry

Major Components of the Ambient Seismic Spectrum



Longuet-Higgins, M. S. (1950), A theory of the origin of microseisms Bromirshi, Webb, Gerstoft, Tanimoto, etc

Weaver et al.(2001), On the emergence of the Green's function in the correlations of a diffuse field, Acoust. Soc. Am.

Snieder(2002),Coda Wave Interferometry for estimating Nonlinear Behavior in seismic velocity, Science

Campilo et al. (2003), Long-Range Correlations in the Diffuse Seismic Coda, Science

Campilo et al. (2003),Emergence of broadband Rayleigh waves from correlattions of the ambient seismic noise, GRL

Snieder (2004), Extracting the Green's function from the correlation of coda waves: a derivation based on stationary phase, Physical. Reiview E.

Campilo et al. (2004), High-Resolution Surface-Wave Tomography from Ambient Seismic noise, Science

Gutenberg Medal Every Year



Haruo Sato

2018



Hitoshi
Kawakatsu

2017



Roel Snieder

2016



Göran Ekström

2015



Gregory C.
Beroza

2014



Jeroen Tromp

2013



Michel Campillo

2012



Guy Masters

2011



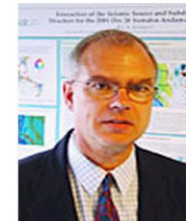
Jean Paul
Montagner

2010



John
Woodhouse

2008



Brian L.N.
Kennett

2007



Guust Nolet

2006



Keiiti Aki

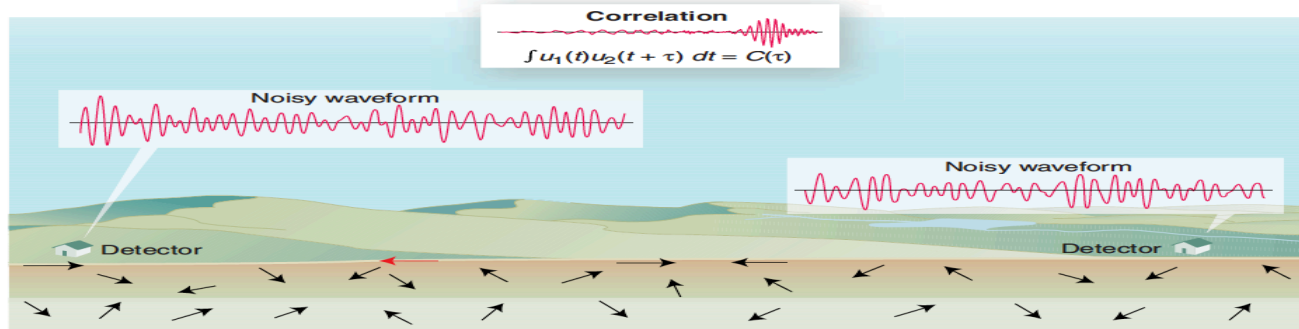
2005



Lev Vinnik

2004

2. Extracting green function from ambient noise



Weaver, 2005, Science

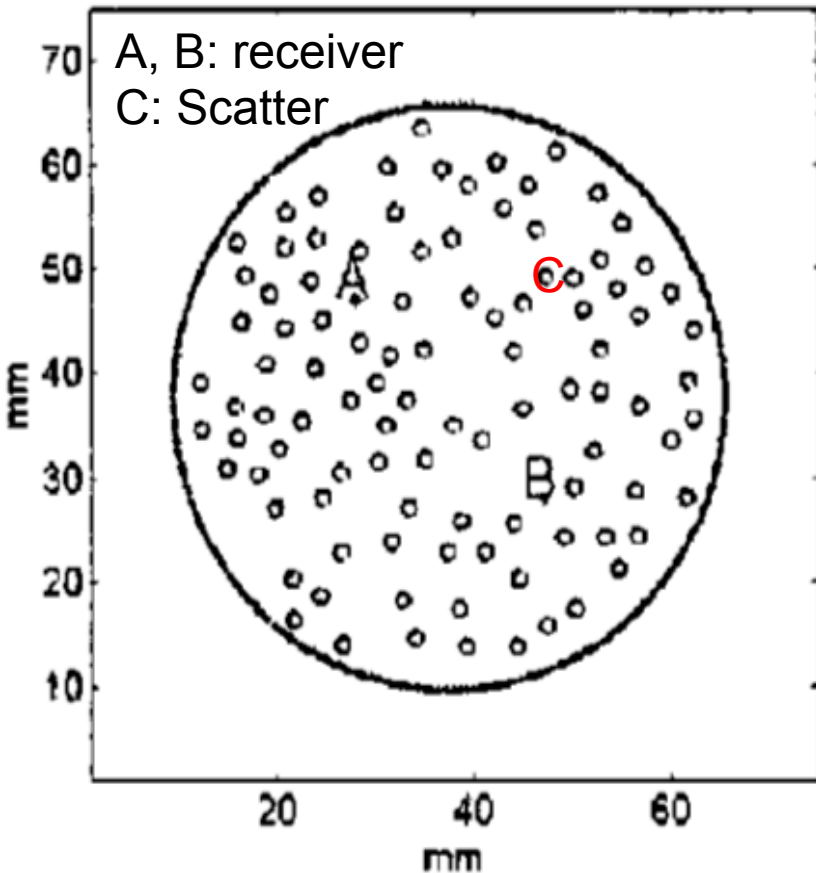
‘noise’ cross-correlation
function

$$\frac{dC_{AB}(t)}{dt} = -\hat{G}_{AB}(t) + \hat{G}_{BA}(-t) \approx -G_{AB}(t) + G_{BA}(-t)$$

Empirical Green’s function (EGF) from
a ‘source’ at A and a receiver at B

Real Green’s function from
a ‘source’ at A and a receiver at B

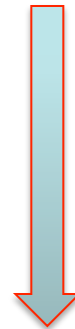
2. Extracting green function from ambient noise



Derode et al (2003).

$$C_{AB}(t) = \int u_B(\tau+t) u_A(\tau) d\tau = G_{AC}(-t) \otimes G_{BC}(t) \otimes f(t)$$

$$f(t) = e(-t) \otimes e(t)$$



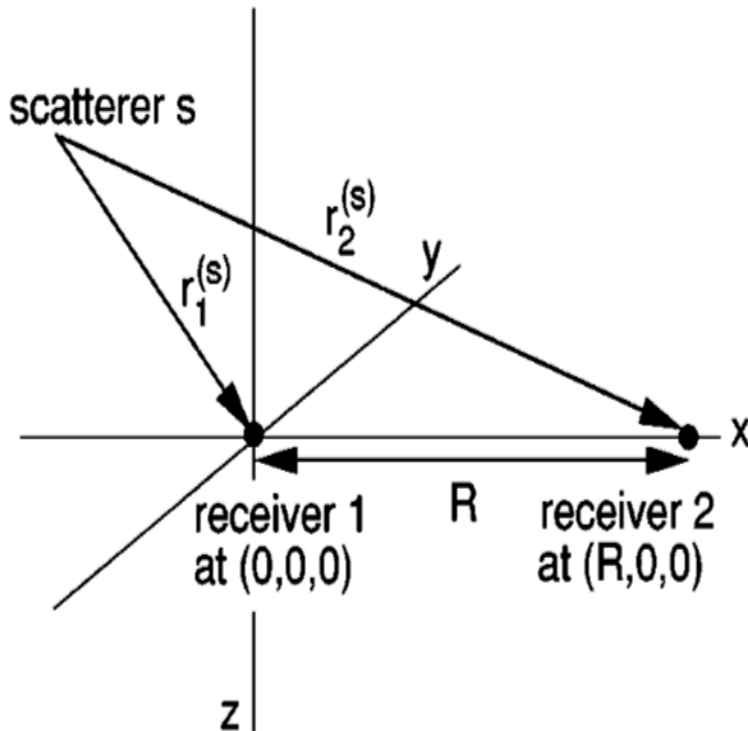
?

$$G_{AB}(t)$$

$$\sum_C G_{AC}(-t) \otimes G_{CB}(t) = G_{AB}(t) + G_{AB}(-t)$$

Like Feymann integral and Kirchhoff theorem

2. Extracting green function from ambient noise



$$C(\tau) = \sum_{s,s'} \int_0^T S_s(t) S_{s'} \left(t + \frac{r_1^{(s')} - r_2^{(s')}}{c} + \tau \right) dt / r_1^{(s')} r_2^{(s')}.$$

$$C(\tau) = \sum_s C_s \left(\tau + \frac{r_1^{(s)} - r_2^{(s)}}{c} \right) / r_1^{(s)} r_2^{(s)}.$$

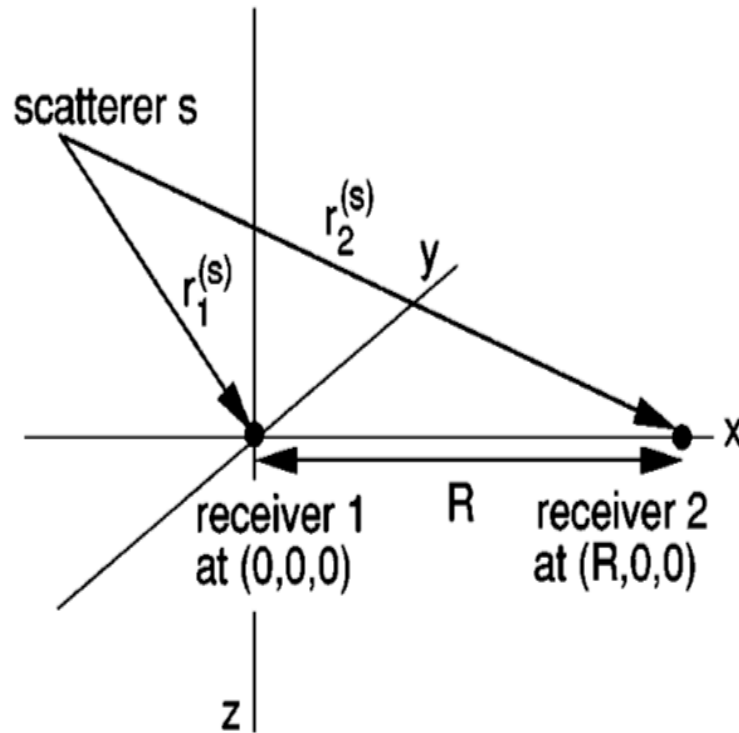
$$C(\omega) = \sum_s |S_s(\omega)|^2 \frac{\exp[i\omega(r_2^{(s)} - r_1^{(s)})/c]}{r_1^{(s)} r_2^{(s)}}.$$

$$C(\omega) = \overline{|S(\omega)|^2} \sum_s \frac{\exp[i\omega(r_2^{(s)} - r_1^{(s)})/c]}{r_1^{(s)} r_2^{(s)}},$$

$$C(\omega) = \overline{|S(\omega)|^2} \int \frac{\exp[i\omega(r_2 - r_1)/c]}{r_1 r_2} n dx dy dz,$$

Snieder et al (2004).

2. Extracting green function from ambient noise



$$C(\omega) = \overline{|S(\omega)|^2} \sum_s \frac{\exp[i\omega(r_2^{(s)} - r_1^{(s)})/c]}{r_1^{(s)} r_2^{(s)}},$$

$$C(\omega) = \overline{|S(\omega)|^2} \int \frac{\exp[i\omega(r_2 - r_1)/c]}{r_1 r_2} n dx dy dz,$$

$$C(\omega) = 2\pi \overline{|S(\omega)|^2} \frac{c}{-i\omega} \int_{-\infty}^{\infty} \frac{e^{ik(|R-x|-|x|)}}{|R-x|-|x|} n dx.$$

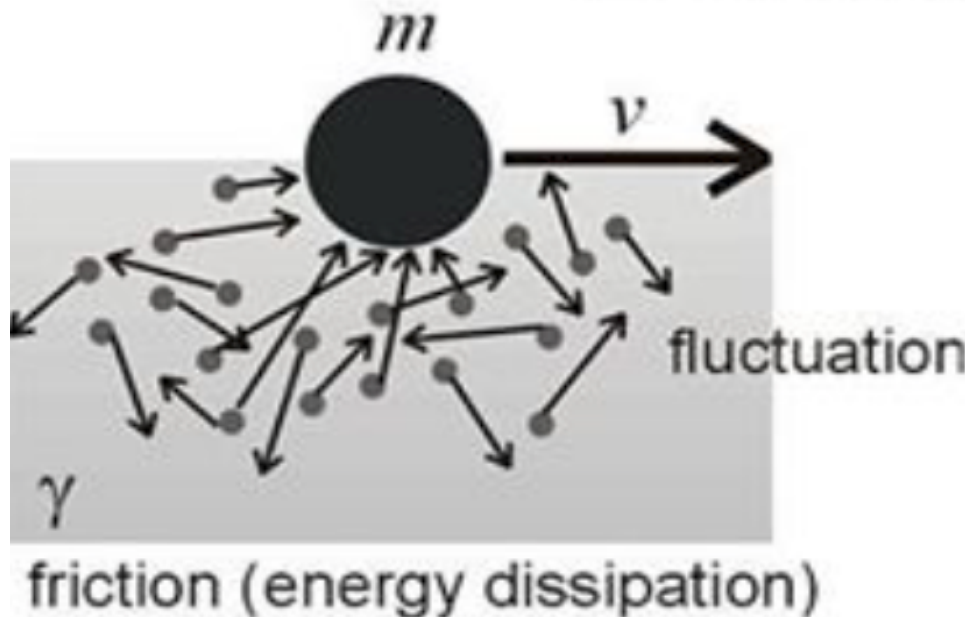
$$C(\omega) = 8\pi^2 \overline{|S(\omega)|^2} \left(\frac{ncL}{i\omega} \right) \left(-\frac{e^{ikR}e^{-R/2L}}{4\pi R} - \frac{e^{-ikR}e^{-R/2L}}{4\pi R} \right).$$

Snieder et al (2004).

2. Extracting green function from ambient noise

Fluctuation–Dissipation Theorem(FDT)

The theory connects the **random fluctuation of linear system** with the **system response** to an external force , Like Brownian motion. **(Shapiro et al., 2005, Science)**



“Forward”: Transfer kinetic energy to heat

“Reverse”: Transfer heat to kinetic energy

That is why Green function can be extracted

3. The principle of noise tomography

$$\left. \begin{aligned} \rho \partial_t^2 s - \nabla \cdot T &= f \\ n \cdot T &= 0 \\ T &= c : \nabla s \end{aligned} \right\}$$

$$C_{ijkl} = (\kappa - \frac{2}{3}\mu)\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$\begin{aligned} T_{ij} &= C_{ijkl} \partial_k s_l = (\kappa - \frac{2}{3}\mu)\delta_{ij} \partial_k s_k + \mu(\partial_i s_j + \partial_j s_i) \\ &= (\kappa - \frac{2}{3}\mu)\delta_{ij} \theta + 2\mu \varepsilon_{ij} \end{aligned}$$

$$T = \kappa \theta I + 2\mu(\varepsilon - \frac{1}{3}(\text{tr} \varepsilon)I)$$

$$T = \frac{1}{3}(\text{tr} T)I + [T - \frac{1}{3}(\text{tr} T)I]$$

Isotropic and deviatoric part

Variable	Isotropic	Deviatoric
Strain ε	$\theta = \text{tr} \varepsilon$	$d = \varepsilon - \frac{1}{3}(\text{tr} \varepsilon)I$
Stress σ	$-p = \frac{1}{3} \text{tr} T$	$\tau = T - \frac{1}{3}(\text{tr} T)I$
Modulus M	κ	2μ

3. The principle of noise tomography

$$s^\alpha(t) \equiv \hat{\mathbf{v}}^\alpha \cdot \mathbf{s}(\mathbf{x}^\alpha, t), \quad s^\beta(t) \equiv \hat{\mathbf{v}}^\beta \cdot \mathbf{s}(\mathbf{x}^\beta, t).$$

Let us consider the $\hat{\mathbf{v}}^\alpha$ component of the displacement at location \mathbf{x}^α ,

the $\hat{\mathbf{v}}^\beta$ component of the displacement at location \mathbf{x}^β

$$C^{\alpha\beta}(t) = \frac{1}{2\pi} \int s^\alpha(\omega) s^{\beta*}(\omega) \exp(i\omega t) d\omega,$$

$$s^\alpha(\omega) = \hat{\mathbf{v}}^\alpha \cdot \mathbf{s}(\mathbf{x}^\alpha, \omega) = \hat{v}_i^\alpha \int G_{ij}(\mathbf{x}^\alpha, \mathbf{x}'; \omega) f_j(\mathbf{x}', \omega) d^3 \mathbf{x}'. \quad \text{Only one source!}$$

$$C^{\alpha\beta}(t) = \frac{1}{2\pi} \hat{v}_i^\alpha \hat{v}_\ell^\beta \iiint G_{ij}(\mathbf{x}^\alpha, \mathbf{x}'; \omega) f_j(\mathbf{x}', \omega) G_{\ell m}^*(\mathbf{x}^\beta, \mathbf{x}''; \omega) f_m^*(\mathbf{x}'', \omega) \exp(i\omega t) d^3 \mathbf{x}' d^3 \mathbf{x}'' d\omega.$$

$$\langle f_j(\mathbf{x}', t') f_m(\mathbf{x}'', t'') \rangle = S_{jm}(\mathbf{x}', t' - t'') \delta(\mathbf{x}' - \mathbf{x}''),$$

Noise generated on surface, 9 components

3. The principle of noise tomography

$$\langle f_j(\mathbf{x}', t') f_m(\mathbf{x}'', t'') \rangle = S_{jm}(\mathbf{x}', t' - t'') \delta(\mathbf{x}' - \mathbf{x}''), \quad \text{Noise generated on surface}$$

$$\langle f_j(x', t') f_m(x'', t'') \rangle = E[f_j(x', t') f_m(x'', t'')]$$

We can extract S from the power spectrum of ensemble average noise. Or we seek the central limit of large samples of the cross correlation

$$\langle C^{\alpha\beta} \rangle(t) = \frac{1}{2\pi} \hat{v}_i^\alpha \hat{v}_\ell^\beta \iint S_{jm}(\mathbf{x}, \omega) G_{ji}(\mathbf{x}, \mathbf{x}^\alpha; \omega) G_{m\ell}^*(\mathbf{x}, \mathbf{x}^\beta; \omega) \exp(i\omega t) d^3\mathbf{x} d\omega.$$

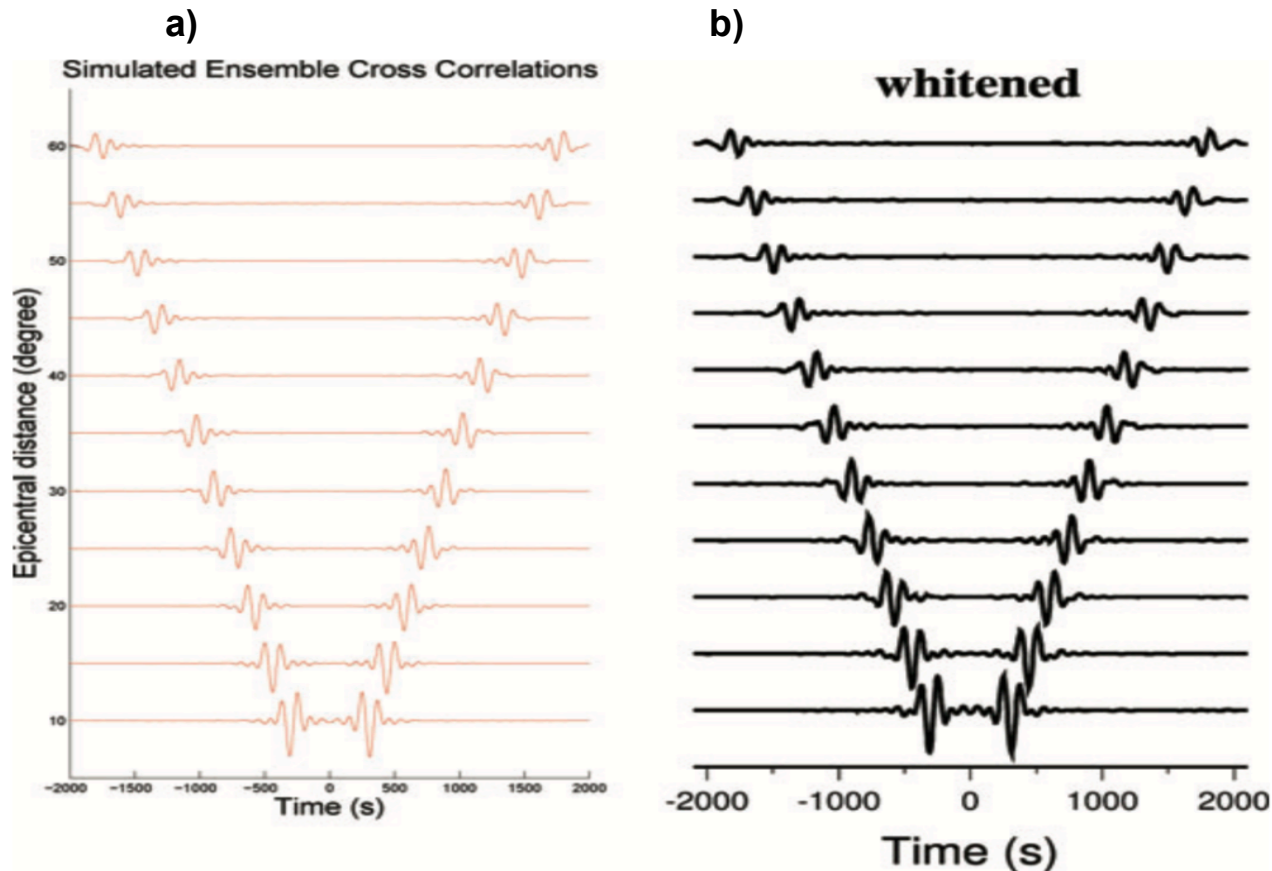
Need note two operations! One is in regular time, the other is in time-reverse

$$\langle C^{\alpha\beta} \rangle(t) = \langle C^{\beta\alpha} \rangle(-t).$$

One usually takes the option of measuring the positive branches ($t > 0$) of $\langle C^{\alpha\beta} \rangle(t)$ and $\langle C^{\beta\alpha} \rangle(t)$,

Symmetric can also facilitate noise cross correlation tomography

3. The principle of noise tomography



a) Ensemble cross correlation obtained based on two numerical simulations Of 7hr global seismic wave propagation

b) Ensemble cross correlation obtained by stacking 5000 synthetic cross correlation

3. The principle of noise tomography

Objective function (OF):

$$\chi = \frac{1}{2} \int [\langle C^{\alpha\beta} \rangle_{\text{sim}} - \langle C^{\alpha\beta} \rangle_{\text{obs}}]^2 dt - \left\langle \iint_{\Omega} \lambda \cdot (\rho \partial_t^2 \mathbf{s} - \nabla \cdot \mathbf{T} - \mathbf{f}) d^3 \mathbf{x} dt \right\rangle$$

Making variation for misfit function $(\langle C^{\alpha\beta} \rangle_{\text{obs}}, \rho, \mathbf{c}, \mathbf{s}, \mathbf{f})$:

$$\begin{aligned} \delta \chi = & \int \Delta \langle C^{\alpha\beta} \rangle \delta \langle C^{\alpha\beta} \rangle dt - \left\langle \iint_{\Omega} \lambda \cdot [\delta \rho \partial_t^2 \mathbf{s} - \nabla \cdot (\delta \mathbf{c} : \nabla \mathbf{s}) - \delta \mathbf{f}] d^3 \mathbf{x} dt \right\rangle \\ & - \left\langle \iint_{\Omega} \lambda \cdot [\rho \partial_t^2 \delta \mathbf{s} - \nabla \cdot (\mathbf{c} : \nabla \delta \mathbf{s})] d^3 \mathbf{x} dt \right\rangle, \end{aligned}$$

Plus boundary condition and Tensor Green Theorem:

$$\hat{\mathbf{n}} \cdot (\delta \mathbf{c} : \nabla \mathbf{s} + \mathbf{c} : \nabla \delta \mathbf{s}) = \mathbf{0} \text{ on } \partial \Omega.$$

$$\begin{aligned} \delta \chi = & \int \Delta \langle C^{\alpha\beta} \rangle \delta \langle C^{\alpha\beta} \rangle dt - \left\langle \iint_{\Omega} (\delta \rho \lambda \cdot \partial_t^2 \mathbf{s} + \nabla \lambda : \delta \mathbf{c} : \nabla \mathbf{s} - \lambda \cdot \delta \mathbf{f}) d^3 \mathbf{x} dt \right\rangle \\ & - \left\langle \iint_{\Omega} [\rho \partial_t^2 \lambda - \nabla \cdot (\mathbf{c} : \nabla \lambda)] \cdot \delta \mathbf{s} d^3 \mathbf{x} dt \right\rangle - \left\langle \iint_{\partial \Omega} \hat{\mathbf{n}} \cdot (\mathbf{c} : \nabla \lambda) \cdot \delta \mathbf{s} d^2 \mathbf{x} dt \right\rangle \end{aligned}$$

3. The principle of noise tomography

$$\int \Delta \langle C^{\alpha\beta} \rangle \delta \langle C^{\alpha\beta} \rangle dt = \int \Delta \langle C^{\alpha\beta} \rangle \langle \delta C^{\alpha\beta} \rangle dt = \left\langle \int \Delta \langle C^{\alpha\beta} \rangle \delta C^{\alpha\beta} dt \right\rangle$$

$$\delta C^{\alpha\beta}(t) = \int [s^\alpha(t + \tau) \delta s^\beta(\tau) + \delta s^\alpha(t + \tau) s^\beta(\tau)] d\tau.$$

$$\int \Delta \langle C^{\alpha\beta} \rangle \delta \langle C^{\alpha\beta} \rangle dt = \left\langle \int \Delta \langle C^{\alpha\beta} \rangle \left[\int [s^\alpha(t + \tau) \delta s^\beta(\tau) + \delta s^\alpha(t + \tau) s^\beta(\tau)] d\tau \right] dt \right\rangle$$

The first terms on the right hand:

$$\iiint \Delta \langle C^{\alpha\beta} \rangle(\tau) s^\alpha(t + \tau) d\tau \delta(\mathbf{x} - \mathbf{x}^\beta) \hat{\mathbf{v}}^\beta \cdot \delta \mathbf{s}(\mathbf{x}, t) d^3 \mathbf{x} dt$$

The Second terms on the right hand:

$$\iiint \Delta \langle C^{\alpha\beta} \rangle(t) s^\beta(\tau) \delta(\mathbf{x} - \mathbf{x}^\alpha) \hat{\mathbf{v}}^\alpha \cdot \delta \mathbf{s}(\mathbf{x}, t + \tau) d^3 \mathbf{x} d\tau dt.$$

3. The principle of noise tomography

The first terms on the right hand

$$\iiint \Delta \langle C^{\alpha\beta} \rangle(\tau) s^\alpha(t + \tau) d\tau \delta(\mathbf{x} - \mathbf{x}^\beta) \hat{\mathbf{v}}^\beta \cdot \delta \mathbf{s}(\mathbf{x}, t) d^3 \mathbf{x} dt$$

The Second terms on the right hand

$$\iiint \Delta \langle C^{\alpha\beta} \rangle(t) s^\beta(\tau) \delta(\mathbf{x} - \mathbf{x}^\alpha) \hat{\mathbf{v}}^\alpha \cdot \delta \mathbf{s}(\mathbf{x}, t + \tau) d^3 \mathbf{x} d\tau dt.$$

Define $t' = t + \tau$, that is, $\tau = t' - t$:

$$\begin{aligned} & \iiint \Delta \langle C^{\alpha\beta} \rangle(t) s^\beta(t' - t) dt \delta(\mathbf{x} - \mathbf{x}^\alpha) \hat{\mathbf{v}}^\alpha \cdot \delta \mathbf{s}(\mathbf{x}, t') d^3 \mathbf{x} dt' \\ &= \iiint \Delta \langle C^{\alpha\beta} \rangle(\tau) s^\beta(t - \tau) d\tau \delta(\mathbf{x} - \mathbf{x}^\alpha) \hat{\mathbf{v}}^\alpha \cdot \delta \mathbf{s}(\mathbf{x}, t) d^3 \mathbf{x} dt \\ &= \iiint \Delta \langle C^{\alpha\beta} \rangle(-\tau) s^\beta(t + \tau) d\tau \delta(\mathbf{x} - \mathbf{x}^\alpha) \hat{\mathbf{v}}^\alpha \cdot \delta \mathbf{s}(\mathbf{x}, t) d^3 \mathbf{x} dt \\ &= \iiint \Delta \langle C^{\beta\alpha} \rangle(\tau) s^\beta(t + \tau) d\tau \delta(\mathbf{x} - \mathbf{x}^\alpha) \hat{\mathbf{v}}^\alpha \cdot \delta \mathbf{s}(\mathbf{x}, t) d^3 \mathbf{x} dt, \end{aligned}$$

$$\langle g, f_1 \oplus f_2 \rangle = \langle f_1 \oplus g, f_2 \rangle$$

3. The principle of noise tomography

$$\int \Delta \langle C^{\alpha\beta} \rangle \delta \langle C^{\alpha\beta} \rangle dt = \left\langle \iint \left[\hat{\mathbf{v}}^\beta \int \Delta \langle C^{\alpha\beta} \rangle(\tau) s^\alpha(t + \tau) d\tau \delta(\mathbf{x} - \mathbf{x}^\beta) + \hat{\mathbf{v}}^\alpha \int \Delta \langle C^{\beta\alpha} \rangle(\tau) s^\beta(t + \tau) d\tau \delta(\mathbf{x} - \mathbf{x}^\alpha) \right] \cdot \delta \mathbf{s}(\mathbf{x}, t) d^3 \mathbf{x} dt \right\rangle.$$

Previous formula:

$$\begin{aligned} \delta \chi = & \int \Delta \langle C^{\alpha\beta} \rangle \delta \langle C^{\alpha\beta} \rangle dt - \left\langle \iint_{\Omega} (\delta \rho \boldsymbol{\lambda} \cdot \partial_t^2 \mathbf{s} + \nabla \boldsymbol{\lambda} : \delta \mathbf{c} : \nabla \mathbf{s} - \boldsymbol{\lambda} \cdot \delta \mathbf{f}) d^3 \mathbf{x} dt \right\rangle \\ & - \left\langle \iint_{\Omega} [\rho \partial_t^2 \boldsymbol{\lambda} - \nabla \cdot (\mathbf{c} : \nabla \boldsymbol{\lambda})] \cdot \delta \mathbf{s} d^3 \mathbf{x} dt \right\rangle - \left\langle \iint_{\partial \Omega} \hat{\mathbf{n}} \cdot (\mathbf{c} : \nabla \boldsymbol{\lambda}) \cdot \delta \mathbf{s} d^2 \mathbf{x} dt \right\rangle \end{aligned}$$

the variation in the action (B2) is stationary with respect to perturbations $\delta \mathbf{s}$ provided the Lagrange multiplier $\boldsymbol{\lambda}$ satisfies the equation

$$\rho \partial_t^2 \boldsymbol{\lambda} - \nabla \cdot (\mathbf{c} : \nabla \boldsymbol{\lambda}) = \hat{\mathbf{v}}^\beta \int \Delta \langle C^{\alpha\beta} \rangle(\tau) s^\alpha(t + \tau) d\tau \delta(\mathbf{x} - \mathbf{x}^\beta) + \hat{\mathbf{v}}^\alpha \int \Delta \langle C^{\beta\alpha} \rangle(\tau) s^\beta(t + \tau) d\tau \delta(\mathbf{x} - \mathbf{x}^\alpha),$$

$$\hat{\mathbf{n}} \cdot (\mathbf{c} : \nabla \boldsymbol{\lambda}) = \mathbf{0} \quad \text{on } \partial \Omega.$$

3. The principle of noise tomography

First define "adjoint wavefield \mathbf{s}^\dagger ":

$$\mathbf{s}^\dagger(\mathbf{x}, t) \equiv \boldsymbol{\lambda}(\mathbf{x}, -t).$$

that is, the adjoint wavefield is the time-reversed Lagrange multiplier wavefield $\boldsymbol{\lambda}$

Then the adjoint wavefield \mathbf{s}^\dagger is determined by

$$\rho \partial_t^2 \mathbf{s}^\dagger - \nabla \cdot \mathbf{T}^\dagger = \mathbf{f}^\dagger,$$

where \mathbf{f}^\dagger denotes the 'adjoint source'

$$\mathbf{f}^\dagger(\mathbf{x}, t) = \hat{\mathbf{v}}^\beta \int \Delta \langle C^{\alpha\beta} \rangle(\tau) s^\alpha(-t + \tau) d\tau \delta(\mathbf{x} - \mathbf{x}^\beta) + \hat{\mathbf{v}}^\alpha \int \Delta \langle C^{\beta\alpha} \rangle(\tau) s^\beta(-t + \tau) d\tau \delta(\mathbf{x} - \mathbf{x}^\alpha),$$

$$\hat{\mathbf{n}} \cdot \mathbf{T}^\dagger = \mathbf{0} \quad \text{on } \partial\Omega.$$

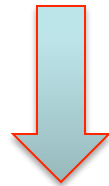
we see that the adjoint wavefield \mathbf{s}^\dagger is determined by exactly the same wave equation and boundary conditions as the regular wavefield, with the exception of the source term:

the regular wavefield is determined by the distributed noise source

\mathbf{f} , whereas the adjoint wavefield is generated at locations \mathbf{x}^α and \mathbf{x}^β by the adjoint source

3. The principle of noise tomography

$$\begin{aligned} \delta\chi = & \int \Delta \langle C^{\alpha\beta} \rangle \delta \langle C^{\alpha\beta} \rangle dt - \left\langle \iint_{\Omega} (\delta\rho \boldsymbol{\lambda} \cdot \partial_t^2 \mathbf{s} + \nabla \boldsymbol{\lambda} : \delta \mathbf{c} : \nabla \mathbf{s} - \boldsymbol{\lambda} \cdot \delta \mathbf{f}) d^3 \mathbf{x} dt \right\rangle \\ & - \left\langle \iint_{\Omega} [\rho \partial_t^2 \boldsymbol{\lambda} - \nabla \cdot (\mathbf{c} : \nabla \boldsymbol{\lambda})] \cdot \delta \mathbf{s} d^3 \mathbf{x} dt \right\rangle - \left\langle \iint_{\partial\Omega} \hat{\mathbf{n}} \cdot (\mathbf{c} : \nabla \boldsymbol{\lambda}) \cdot \delta \mathbf{s} d^2 \mathbf{x} dt \right\rangle \end{aligned}$$



$$\mathbf{s}^\dagger(\mathbf{x}, t) \equiv \boldsymbol{\lambda}(\mathbf{x}, -t),$$

$$\delta\chi = \left\langle \int_{\Omega} (\delta \ln \rho K_\rho + \delta \mathbf{c} :: \mathbf{K}_c + \delta \Sigma) d^3 \mathbf{x} \right\rangle,$$

where we use the notation $\delta \mathbf{c} :: \mathbf{K}_c = \delta c_{ijkl} K_{cijkl}$. The kernels K_ρ and \mathbf{K}_c are defined by

$$K_\rho \equiv - \int \rho \mathbf{s}^\dagger(-t) \cdot \partial_t^2 \mathbf{s}(t) dt,$$

$$\mathbf{K}_c \equiv - \int \nabla \mathbf{s}^\dagger(-t) \nabla \mathbf{s}(t) dt$$

and

$$\delta \Sigma = \int \mathbf{s}^\dagger \cdot \delta \mathbf{f} dt.$$

3. The principle of noise tomography

$$\delta\chi = \left\langle \int_{\Omega} (\delta \ln \rho K_{\rho} + \delta \mathbf{c} :: \mathbf{K}_{\mathbf{c}} + \delta \Sigma) d^3 \mathbf{x} \right\rangle,$$

where we use the notation $\delta \mathbf{c} :: \mathbf{K}_{\mathbf{c}} = \delta c_{ijkl} K_{cijkl}$. The kernels K_{ρ} and $\mathbf{K}_{\mathbf{c}}$ are defined by

$$K_{\rho} \equiv - \int \rho \mathbf{s}^{\dagger}(-t) \cdot \partial_t^2 \mathbf{s}(t) dt,$$

$$\mathbf{K}_{\mathbf{c}} \equiv - \int \nabla \mathbf{s}^{\dagger}(-t) \nabla \mathbf{s}(t) dt$$

and

$$\delta \Sigma = \int \mathbf{s}^{\dagger} \cdot \delta \mathbf{f} dt.$$

As it stands, eq. (20) is not useable because it involves an ensemble average $\langle \cdot \rangle$. *********

$$\delta\chi = -\frac{1}{2\pi} \left\langle \iiint (-\omega^2 \delta \rho \mathbf{s}^{\dagger} \cdot \mathbf{s} + \nabla \mathbf{s}^{\dagger} : \delta \mathbf{c} : \nabla \mathbf{s} - \mathbf{s}^{\dagger} \cdot \delta \mathbf{f}) d^3 \mathbf{x} d\omega \right\rangle.$$

In the frequency domain, the adjoint source (33) becomes

$$\mathbf{f}^{\dagger}(\mathbf{x}, \omega) = [\hat{\mathbf{v}}^{\beta} \Delta \langle C^{\alpha\beta} \rangle(\omega) s^{\alpha*}(\omega) \delta(\mathbf{x} - \mathbf{x}^{\beta}) + \hat{\mathbf{v}}^{\alpha} \Delta \langle C^{\beta\alpha} \rangle(\omega) s^{\beta*}(\omega) \delta(\mathbf{x} - \mathbf{x}^{\alpha})].$$

Thus the adjoint wavefield is

$$\begin{aligned} \mathbf{s}^{\dagger}(\mathbf{x}, \omega) &= \int \mathbf{G}(\mathbf{x}, \mathbf{x}'; \omega) \cdot \mathbf{f}^{\dagger}(\mathbf{x}', \omega) d^3 \mathbf{x}' \\ &= [\Delta \langle C^{\alpha\beta} \rangle(\omega) \hat{\mathbf{v}}^{\alpha} \cdot \mathbf{s}^*(\mathbf{x}^{\alpha}, \omega) \mathbf{G}(\mathbf{x}, \mathbf{x}^{\beta}; \omega) \cdot \hat{\mathbf{v}}^{\beta} + \Delta \langle C^{\beta\alpha} \rangle(\omega) \hat{\mathbf{v}}^{\beta} \cdot \mathbf{s}^*(\mathbf{x}^{\beta}, \omega) \mathbf{G}(\mathbf{x}, \mathbf{x}^{\alpha}; \omega) \cdot \hat{\mathbf{v}}^{\alpha}]. \end{aligned}$$

3. The principle of noise tomography

$$\begin{aligned}
 \mathbf{s}^\dagger(\mathbf{x}, \omega) &= \int \mathbf{G}(\mathbf{x}, \mathbf{x}'; \omega) \cdot \mathbf{f}^\dagger(\mathbf{x}', \omega) d^3 \mathbf{x}' \\
 &= [\Delta \langle C^{\alpha\beta} \rangle(\omega) \hat{\mathbf{v}}^\alpha \cdot \mathbf{s}^*(\mathbf{x}^\alpha, \omega) \mathbf{G}(\mathbf{x}, \mathbf{x}^\beta; \omega) \cdot \hat{\mathbf{v}}^\beta + \Delta \langle C^{\beta\alpha} \rangle(\omega) \hat{\mathbf{v}}^\beta \cdot \mathbf{s}^*(\mathbf{x}^\beta, \omega) \mathbf{G}(\mathbf{x}, \mathbf{x}^\alpha; \omega) \cdot \hat{\mathbf{v}}^\alpha]. \\
 \delta\chi &= -\frac{1}{2\pi} \left\langle \iint (-\omega^2 \delta\rho s_i^\dagger s_i + \nabla_i s_j^\dagger \delta c_{ijkl} \nabla_k s_\ell - s_i^\dagger \delta f_i) d^3 \mathbf{x} d\omega \right\rangle \\
 &= -\frac{1}{2\pi} \left\langle \iiint \left\{ -\omega^2 \delta\rho [\Delta \langle C^{\alpha\beta} \rangle \hat{\mathbf{v}}_m^\alpha G_{mp}^*(\mathbf{x}^\alpha, \mathbf{x}') G_{in}(\mathbf{x}, \mathbf{x}^\beta) \hat{\mathbf{v}}_n^\beta \right. \right. \\
 &\quad + \Delta \langle C^{\beta\alpha} \rangle \hat{\mathbf{v}}_m^\beta G_{mp}^*(\mathbf{x}^\beta, \mathbf{x}') G_{in}(\mathbf{x}, \mathbf{x}^\alpha) \hat{\mathbf{v}}_n^\alpha] G_{iq}(\mathbf{x}, \mathbf{x}'') \\
 &\quad + \delta c_{ijkl} [\Delta \langle C^{\alpha\beta} \rangle \hat{\mathbf{v}}_m^\alpha G_{mp}^*(\mathbf{x}^\alpha, \mathbf{x}') \nabla_i G_{jn}(\mathbf{x}, \mathbf{x}^\beta) \hat{\mathbf{v}}_n^\beta \\
 &\quad + \Delta \langle C^{\beta\alpha} \rangle \hat{\mathbf{v}}_m^\beta G_{mp}^*(\mathbf{x}^\beta, \mathbf{x}') \nabla_i G_{jn}(\mathbf{x}, \mathbf{x}^\alpha) \hat{\mathbf{v}}_n^\alpha] \nabla_k G_{\ell q}(\mathbf{x}, \mathbf{x}'') \left. \right\} f_p^*(\mathbf{x}') f_q(\mathbf{x}'') d^3 \mathbf{x} d^3 \mathbf{x}' d^3 \mathbf{x}'' d\omega \right\rangle \\
 &\quad + \frac{1}{2\pi} \left\langle \iiint [\Delta \langle C^{\alpha\beta} \rangle \hat{\mathbf{v}}_m^\alpha G_{mp}^*(\mathbf{x}^\alpha, \mathbf{x}') G_{in}(\mathbf{x}, \mathbf{x}^\beta) \hat{\mathbf{v}}_n^\beta \right. \\
 &\quad \left. + \Delta \langle C^{\beta\alpha} \rangle \hat{\mathbf{v}}_m^\beta G_{mp}^*(\mathbf{x}^\beta, \mathbf{x}') G_{in}(\mathbf{x}, \mathbf{x}^\alpha) \hat{\mathbf{v}}_n^\alpha] f_p^*(\mathbf{x}') \delta f_i(\mathbf{x}) d^3 \mathbf{x} d^3 \mathbf{x}' d\omega \right\rangle.
 \end{aligned}$$

3. The principle of noise tomography

Now we can take the desired ensemble average to obtain

$$\begin{aligned}
 \delta\chi = & -\frac{1}{2\pi} \iiint \left\{ -\omega^2 \delta\rho \left[\Delta \langle C^{\alpha\beta} \rangle \hat{v}_m^\alpha G_{mp}^*(\mathbf{x}^\alpha, \mathbf{x}') G_{in}(\mathbf{x}, \mathbf{x}^\beta) \hat{v}_n^\beta \right. \right. \\
 & + \Delta \langle C^{\beta\alpha} \rangle \hat{v}_m^\beta G_{mp}^*(\mathbf{x}^\beta, \mathbf{x}') G_{in}(\mathbf{x}, \mathbf{x}^\alpha) \hat{v}_n^\alpha \left. \right] G_{iq}(\mathbf{x}, \mathbf{x}') \\
 & + \delta c_{ijkl} \left[\Delta \langle C^{\alpha\beta} \rangle \hat{v}_m^\alpha G_{mp}^*(\mathbf{x}^\alpha, \mathbf{x}') \nabla_i G_{jn}(\mathbf{x}, \mathbf{x}^\beta) \hat{v}_n^\beta \right. \\
 & + \Delta \langle C^{\beta\alpha} \rangle \hat{v}_m^\beta G_{mp}^*(\mathbf{x}^\beta, \mathbf{x}') \nabla_i G_{jn}(\mathbf{x}, \mathbf{x}^\alpha) \hat{v}_n^\alpha \left. \right] \nabla_k G_{\ell q}(\mathbf{x}, \mathbf{x}') \left. \right\} S_{pq}(\mathbf{x}', \omega) d^3\mathbf{x} d^3\mathbf{x}' d\omega \\
 & + \frac{1}{2\pi} \iint \left[\Delta \langle C^{\alpha\beta} \rangle \hat{v}_m^\alpha G_{mp}^*(\mathbf{x}^\alpha, \mathbf{x}) G_{in}(\mathbf{x}, \mathbf{x}^\beta) \hat{v}_n^\beta \right. \\
 & \left. + \Delta \langle C^{\beta\alpha} \rangle \hat{v}_m^\beta G_{mp}^*(\mathbf{x}^\beta, \mathbf{x}) G_{in}(\mathbf{x}, \mathbf{x}^\alpha) \hat{v}_n^\alpha \right] \delta S_{pi}(\mathbf{x}', \omega) d^3\mathbf{x} d\omega,
 \end{aligned}$$

To bring this into a more practical form, let us define the source

$$F_i^\alpha(\mathbf{x}, \omega) \equiv G_{jk}^*(\mathbf{x}, \mathbf{x}^\alpha; \omega) v_k^\alpha S_{ij}(\mathbf{x}, \omega). \quad (\text{in reverse time})$$

The corresponding wavefield is

$$\Phi^\alpha(\mathbf{x}, \omega) \equiv \int_{\Omega} \mathbf{G}(\mathbf{x}, \mathbf{x}'; \omega) \cdot \mathbf{F}^\alpha(\mathbf{x}', \omega) d^3\mathbf{x}'. \quad (\text{The cross-correlation function, regular Field, equivalent correlation with two fields})$$

3. The principle of noise tomography

Now we can take the desired ensemble average to obtain

$$\begin{aligned}
 \delta\chi = & -\frac{1}{2\pi} \iiint \left\{ -\omega^2 \delta\rho \left[\Delta \langle C^{\alpha\beta} \rangle \hat{v}_m^\alpha G_{mp}^*(\mathbf{x}^\alpha, \mathbf{x}') G_{in}(\mathbf{x}, \mathbf{x}^\beta) \hat{v}_n^\beta \right. \right. \\
 & + \Delta \langle C^{\beta\alpha} \rangle \hat{v}_m^\beta G_{mp}^*(\mathbf{x}^\beta, \mathbf{x}') G_{in}(\mathbf{x}, \mathbf{x}^\alpha) \hat{v}_n^\alpha \left. \right] G_{iq}(\mathbf{x}, \mathbf{x}') \\
 & + \delta c_{ijkl} \left[\Delta \langle C^{\alpha\beta} \rangle \hat{v}_m^\alpha G_{mp}^*(\mathbf{x}^\alpha, \mathbf{x}') \nabla_i G_{jn}(\mathbf{x}, \mathbf{x}^\beta) \hat{v}_n^\beta \right. \\
 & + \Delta \langle C^{\beta\alpha} \rangle \hat{v}_m^\beta G_{mp}^*(\mathbf{x}^\beta, \mathbf{x}') \nabla_i G_{jn}(\mathbf{x}, \mathbf{x}^\alpha) \hat{v}_n^\alpha \left. \right] \nabla_k G_{\ell q}(\mathbf{x}, \mathbf{x}') \left. \right\} S_{pq}(\mathbf{x}', \omega) d^3\mathbf{x} d^3\mathbf{x}' d\omega \\
 & + \frac{1}{2\pi} \iint \left[\Delta \langle C^{\alpha\beta} \rangle \hat{v}_m^\alpha G_{mp}^*(\mathbf{x}^\alpha, \mathbf{x}) G_{in}(\mathbf{x}, \mathbf{x}^\beta) \hat{v}_n^\beta \right. \\
 & \left. + \Delta \langle C^{\beta\alpha} \rangle \hat{v}_m^\beta G_{mp}^*(\mathbf{x}^\beta, \mathbf{x}) G_{in}(\mathbf{x}, \mathbf{x}^\alpha) \hat{v}_n^\alpha \right] \delta S_{pi}(\mathbf{x}', \omega) d^3\mathbf{x} d\omega,
 \end{aligned}$$

Next, we define the adjoint source

$$\mathbf{F}^{\dagger\alpha\beta}(\mathbf{x}, \omega) \equiv \hat{\mathbf{v}}^\beta \Delta \langle C^{\alpha\beta} \rangle(\omega) \delta(\mathbf{x} - \mathbf{x}^\beta). \quad (\text{The “adjoint” source at location } \mathbf{x}^\beta)$$

The associated adjoint wavefield is

$$\Phi^{\dagger\alpha\beta}(\mathbf{x}, \omega) \equiv \int \mathbf{G}(\mathbf{x}, \mathbf{x}'; \omega) \cdot \mathbf{F}^{\dagger\alpha\beta}(\mathbf{x}', \omega) d^3\mathbf{x}'. \quad (\text{The “adjoint” field at location } \mathbf{x}^\beta)$$

3. The principle of noise tomography

$$\begin{aligned} \delta\chi = & -\frac{1}{2\pi} \iint \left[-\omega^2 \delta\rho \left(\Phi_i^{\dagger\alpha\beta} \Phi_i^\alpha + \Phi_i^{\dagger\beta\alpha} \Phi_i^\beta \right) + \delta c_{ijkl} \left(\nabla_i \Phi_j^{\dagger\alpha\beta} \nabla_k \Phi_\ell^\alpha + \nabla_i \Phi_j^{\dagger\beta\alpha} \nabla_k \Phi_\ell^\beta \right) \right] d^3\mathbf{x} d\omega \\ & + \frac{1}{2\pi} \iint \left(\Phi_i^{\dagger\alpha\beta} \delta F_i^\alpha + \Phi_i^{\dagger\beta\alpha} \delta F_i^\beta \right) d^3\mathbf{x} d\omega, \end{aligned}$$

where

$$\delta F_i^\alpha(\mathbf{x}, \omega) \equiv G_{jk}^*(\mathbf{x}, \mathbf{x}^\alpha; \omega) v_k^\alpha \delta S_{ij}(\mathbf{x}, \omega).$$

Finally, again using Parseval's theorem, the change in the misfit may be written in the form

$$\delta\chi = \int_{\Omega} (\delta \ln \rho \langle K_\rho \rangle + \delta \mathbf{c} :: \langle \mathbf{K}_c \rangle + \langle \delta \Sigma \rangle) d^3\mathbf{x},$$

Time in reverse

where we have defined the ensemble kernels

$$\langle K_\rho \rangle = - \int \rho \left[\Phi^{\dagger\alpha\beta}(-t) \cdot \partial_t^2 \Phi^\alpha(t) + \Phi^{\dagger\beta\alpha}(-t) \cdot \partial_t^2 \Phi^\beta(t) \right] dt,$$

$$\langle \mathbf{K}_c \rangle = - \int \left[\nabla \Phi^{\dagger\alpha\beta}(-t) \nabla \Phi^\alpha(t) + \nabla \Phi^{\dagger\beta\alpha}(-t) \nabla \Phi^\beta(t) \right] dt,$$

and

$$\langle \delta \Sigma \rangle = \int \left[\Phi^{\dagger\alpha\beta}(-t) \cdot \delta \mathbf{F}^\alpha(t) + \Phi^{\dagger\beta\alpha}(-t) \cdot \delta \mathbf{F}^\beta(t) \right] dt.$$

3. The principle of noise tomography

$$\langle K_\rho \rangle = - \int \rho \left[\underbrace{\Phi^{\dagger\alpha\beta}(-t)}_{\text{red dashed}} \cdot \partial_t^2 \Phi^\alpha(t) + \underbrace{\Phi^{\dagger\beta\alpha}(-t)}_{\text{green dashed}} \cdot \partial_t^2 \Phi^\beta(t) \right] dt,$$

$$\langle \mathbf{K}_c \rangle = - \int \left[\underbrace{\nabla \Phi^{\dagger\alpha\beta}(-t)}_{\text{red dashed}} \nabla \Phi^\alpha(t) + \underbrace{\nabla \Phi^{\dagger\beta\alpha}(-t)}_{\text{green dashed}} \nabla \Phi^\beta(t) \right] dt,$$

and where

$$\langle \delta \Sigma \rangle = \int \left[\underbrace{\Phi^{\dagger\alpha\beta}(-t)}_{\text{red dashed}} \cdot \delta \mathbf{F}^\alpha(t) + \underbrace{\Phi^{\dagger\beta\alpha}(-t)}_{\text{green dashed}} \cdot \delta \mathbf{F}^\beta(t) \right] dt.$$

The calculation of (27)–(29) requires access to two types of wavefields:

Φ^α and Φ^β in regular time, t , and $\Phi^{\dagger\alpha\beta}$ and $\Phi^{\dagger\beta\alpha}$ in reverse

time, $-t$. The ‘ensemble forward wavefield’ Φ^α is generated based upon the source

$$\hat{\mathbf{v}}^\alpha \cdot \Phi^\beta(\mathbf{x}^\alpha, t) = \hat{\mathbf{v}}^\beta \cdot \Phi^\alpha(\mathbf{x}^\beta, -t),$$

Each kernel consists of the contribution of $C^{\alpha\beta}$ and $C^{\beta\alpha}$

3. The principle of noise tomography

Workflow

The following steps are involved in the construction of ensemble event kernels (91)–(95) for terrestrial noise cross-correlation tomography.

- (i) Characterize the ensemble-averaged noise, namely $S_{ij}(\mathbf{x}, \omega)$ in (14).
- (ii) For each point of interest \mathbf{x}^α , calculate the generating wavefields $\eta^\alpha(\mathbf{x}, t)$ based upon eq. (31), and store them as a ‘movie’ ‘at locations where the noise is non-zero’.
- (iii) Generate the source \mathbf{F}^α given by (32), which is needed for the calculation of the ensemble wavefield Φ^α .
- (iv) For all pairs \mathbf{x}^α and \mathbf{x}^β , calculate the theoretical ensemble cross correlation $\langle C^{\alpha\beta} \rangle(t)$ based upon expression (38).
- (v) Determine a measure of the difference between observed and simulated ensemble cross correlations. This could be the waveform difference between simulated and measured positive/negative ensemble cross-correlation branches, traveltime delay measurements between simulated and measured ensemble cross correlations in both the positive and negative branches, etc. This forms the data set.
- (vi) Construct the adjoint source $\mathbf{F}^{\dagger\alpha}$ given by (89), which is needed for the calculation of the ensemble adjoint wavefield $\Phi^{\dagger\alpha}$.
- (vii) Generate the ensemble event kernels (91)–(95) based upon a combined calculation of the wavefields $\Phi^\alpha(\mathbf{x}, t)$ and $\Phi^{\dagger\alpha}(\mathbf{x}, -t)$ (see Liu & Tromp 2006).
- (viii) Add the ensemble event kernels to obtain the gradient of the misfit function.

Based on symmetric properties of regular fields and adjoint fields, we just need three forward modelings

3. The principle of noise tomography

Workflow(Specfem3D)

- Step 1: simulation for generating wavefields

```
SIMULATION_TYPE = 1  
NOISE_TOMOGRAPHY = 1  
SAVE_FORWARD (not used, can be either .true. or .false.)
```

- Step 2: simulation for ensemble forward wavefields

```
SIMULATION_TYPE = 1  
NOISE_TOMOGRAPHY = 2  
SAVE_FORWARD = .true.
```

- Step 3: simulation for ensemble adjoint wavefields and sensitivity kernels

```
SIMULATION_TYPE = 3  
NOISE_TOMOGRAPHY = 3  
SAVE_FORWARD = .false.
```

Note Step 3 is an adjoint simulation, please refer to previous chapters on how to prepare adjoint sources and other necessary files, as well as how adjoint simulations work.

3. The principle of noise tomography

1. Full waveform inversion

$$\chi = \frac{1}{N^{\alpha\beta}} \int [\langle C^{\alpha\beta} \rangle_{\text{sim}} - \langle C^{\alpha\beta} \rangle_{\text{obs}}]^2 dt,$$

where

$$N^{\alpha\beta} = \int \langle C^{\alpha\beta} \rangle_{\text{obs}}^2 dt.$$

This would result in the ensemble adjoint source

$$\mathbf{F}^{\dagger\alpha\beta}(\mathbf{x}, t) = \hat{\mathbf{v}}^\beta \Delta \langle C^{\alpha\beta} \rangle(t) \delta(\mathbf{x} - \mathbf{x}^\beta) / N^{\alpha\beta}.$$

2. Travel time inversion

$$\chi = \frac{1}{2} (\Delta T^{\alpha\beta})^2,$$

which would result in the ensemble adjoint source

$$\mathbf{F}^{\dagger\alpha\beta}(\mathbf{x}, t) = -\hat{\mathbf{v}}^\beta \Delta T^{\alpha\beta} \langle \dot{C}^{\alpha\beta} \rangle_{\text{sim}}(t) \delta(\mathbf{x} - \mathbf{x}^\beta) / N'^{\alpha\beta},$$

$$N'^{\alpha\beta} = \int [\langle \dot{C}^{\alpha\beta} \rangle_{\text{sim}}]^2 dt.$$

4. 2-D examples

$$\rho \partial_t^2 s = \nabla \cdot (\mu \nabla s) + f,$$

$$G(\mathbf{x}, \mathbf{x}'; \omega) = \sum_k (\omega_k^2 - \omega^2)^{-1} s_k(\mathbf{x}) s_k(\mathbf{x}'),$$

or in the time domain

$$G(\mathbf{x}, \mathbf{x}'; t) = \sum_k s_k(\mathbf{x}) s_k(\mathbf{x}') \omega_k^{-1} \sin(\omega_k t) H(t),$$

$$\begin{aligned} \langle C^{\alpha\beta} \rangle(\omega) &= \int G(\mathbf{x}, \mathbf{x}^\alpha; \omega) G^*(\mathbf{x}, \mathbf{x}^\beta; \omega) S(\omega) d^2 \mathbf{x} \\ &= \int \sum_k (\omega_k^2 - \omega^2)^{-1} s_k(\mathbf{x}) s_k(\mathbf{x}^\alpha) \sum_{k'} (\omega_{k'}^2 - \omega^2)^{-1} s_{k'}(\mathbf{x}) s_{k'}(\mathbf{x}^\beta) S(\omega) d^2 \mathbf{x}. \end{aligned}$$

Now we assume that the density ρ is constant, and use mode orthonormality (58):

$$\langle C^{\alpha\beta} \rangle(\omega) = \rho^{-1} \sum_k (\omega_k^2 - \omega^2)^{-2} s_k(\mathbf{x}^\alpha) s_k(\mathbf{x}^\beta) S(\omega).$$

$$\tilde{G}(\mathbf{x}, \mathbf{x}'; \omega) \equiv (2\omega)^{-1} \partial_\omega G(\mathbf{x}, \mathbf{x}'; \omega) = \sum_k (\omega_k^2 - \omega^2)^{-2} s_k(\mathbf{x}) s_k(\mathbf{x}'),$$

which has the time domain expression

$$\tilde{G}(\mathbf{x}, \mathbf{x}'; t) = \frac{1}{\gamma} \sum_k s_k(\mathbf{x}) s_k(\mathbf{x}') \omega_k^{-3} \sin(\omega_k t) H(t).$$

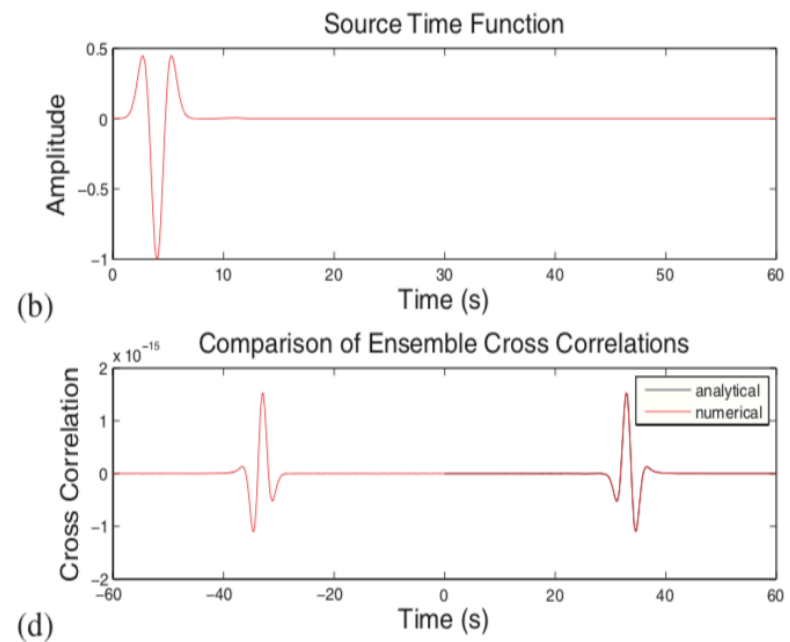
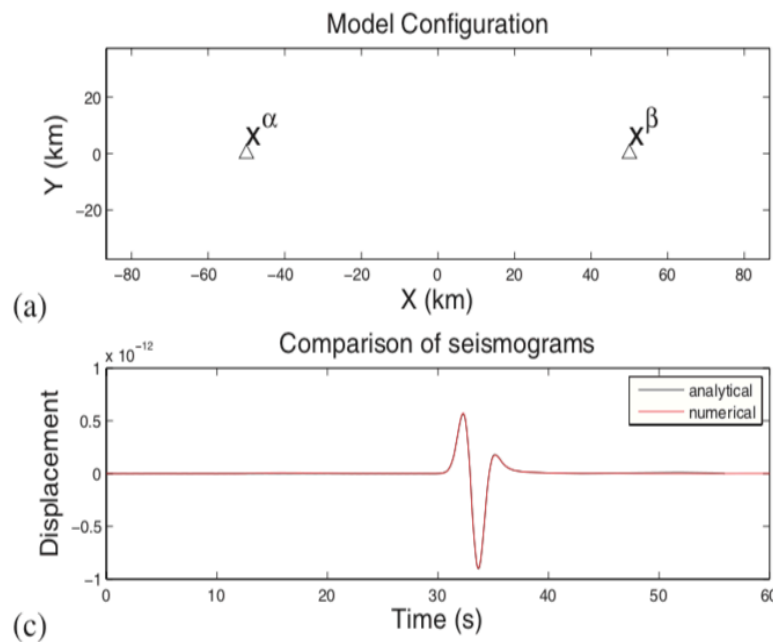
4. 2-D examples

$$\langle C^{\alpha\beta} \rangle(\omega) = \rho^{-1} (2\omega)^{-1} \partial_{\omega} G(\mathbf{x}^{\alpha}, \mathbf{x}^{\beta}; \omega) S(\omega).$$

Then,

$$\Phi^{\alpha}(\mathbf{x}, \omega) = \int G(\mathbf{x}, \mathbf{x}'; \omega) G^{*}(\mathbf{x}', \mathbf{x}^{\alpha}; \omega) S(\omega) d^2 \mathbf{x}' = \rho^{-1} (2\omega)^{-1} \partial_{\omega} G(\mathbf{x}, \mathbf{x}^{\alpha}; \omega) S(\omega),$$

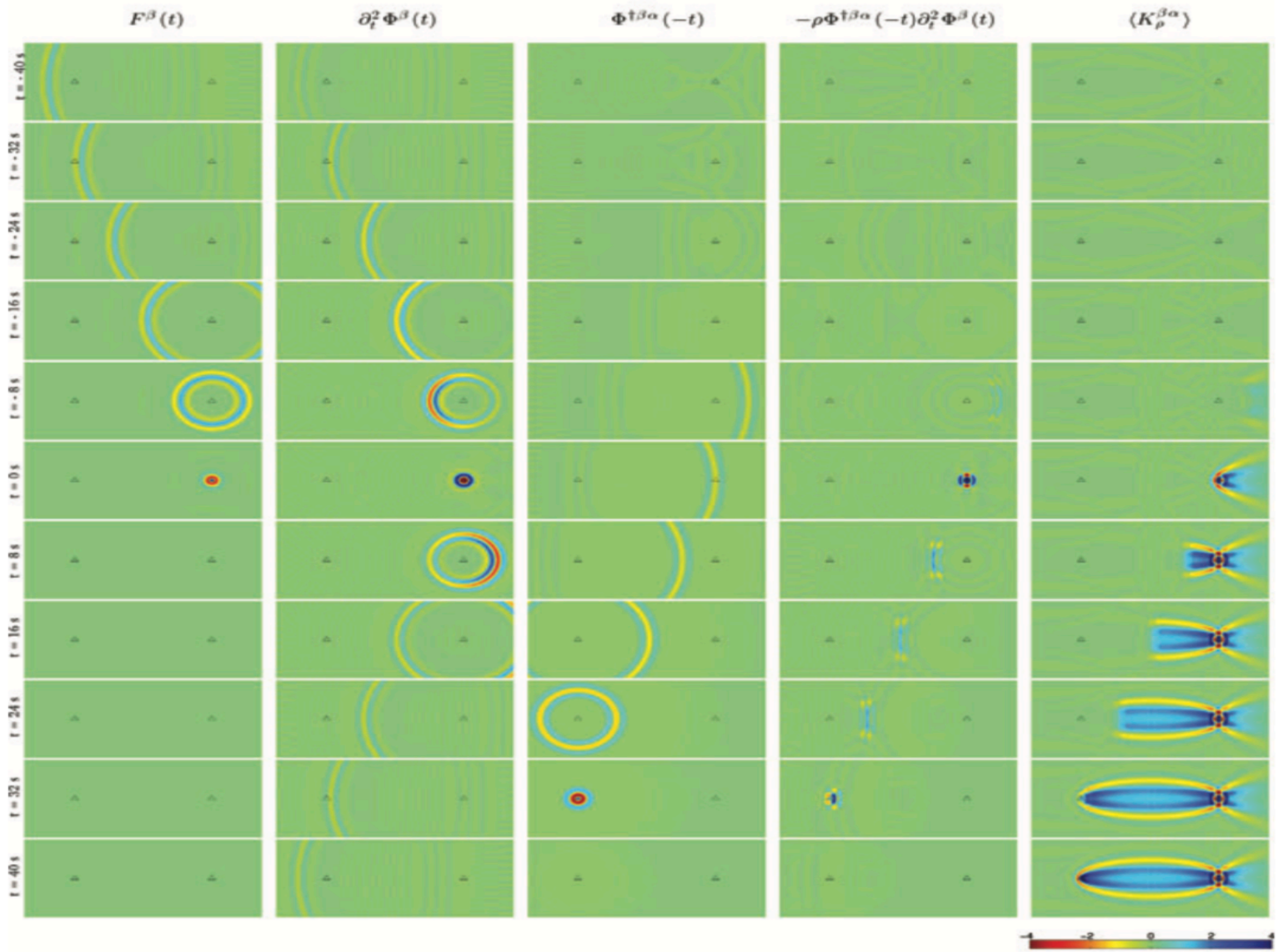
$$\Phi^{\dagger\alpha\beta}(\mathbf{x}, \omega) = G(\mathbf{x}, \mathbf{x}^{\beta}; \omega) \Delta \langle C^{\alpha\beta} \rangle(\omega).$$



4. 2-D examples



4. 2-D examples



3. The principle of noise tomography

Some important operators

$$\begin{array}{ccc}
 \langle m \rangle & & \langle d \rangle \\
 V \xrightleftharpoons[G^*]{G} H & & \hat{V} \text{ is } V\text{'s dual space} \\
 \hat{V} \xrightleftharpoons[G^T]{\hat{G}} \hat{H} & & \hat{H} \text{ is } H\text{'s dual space}
 \end{array}$$

Definition: $G : V \rightarrow H$
 Transpose operator $G^T : \hat{H} \rightarrow \hat{V} : \langle G^T \hat{d}, m \rangle_m = \langle \hat{d}, Gm \rangle_D$
 adjoint operator $G^* : H \rightarrow V : \langle G^* d, m \rangle_m = \langle d, Gm \rangle_D$
 dual operator $\hat{G} : \langle \hat{G} \hat{m}, d \rangle_D = \langle Gm, d \rangle_D$

$$\hat{G} = \langle G^* \rangle^T$$

3. The principle of noise tomography

$$X = \frac{1}{2} \int [\langle C_{\text{syn}}^{\partial B} - C_{\text{obs}}^{\partial B} \rangle^2] dt - \lambda \langle L S - f \rangle$$

$$b_X = \langle \Delta C^{\partial B}, b C_{\text{obs}}^{\partial B} \rangle - \langle \lambda, L b S \rangle - \langle \lambda, b L S \rangle + \langle \lambda, b f \rangle$$

$$\langle \lambda, L b S \rangle - \langle L^* \lambda, b S \rangle = B_1 \langle \lambda, b S \rangle \quad (1)$$

$$L = \rho \partial_t^2 + \nabla \cdot (C : \nabla) \Rightarrow b L = \delta \rho \partial_t^2 + \nabla \cdot (b C : \nabla)$$

$$\langle \lambda, b L S \rangle = \langle \lambda, \delta \rho \partial_t^2 S + \nabla \cdot (b C : \nabla S) \rangle$$

$$= \langle \lambda, b \rho \partial_t^2 S \rangle + \langle \lambda, \nabla \cdot (b C : \nabla S) \rangle$$

$$= \langle \lambda, b \rho \partial_t^2 S \rangle + \langle \nabla \lambda, b C : \nabla S \rangle + B_2(b C, \lambda) \quad (2)$$

$$b_X = \langle \Delta C^{\partial B}, b C_{\text{obs}}^{\partial B} \rangle - \underbrace{\langle L^* \lambda, b S \rangle - B_1 \langle \lambda, b S \rangle}_{(1)} + \langle \lambda, b f \rangle$$

3. The principle of noise tomography

$$\begin{aligned}
 x &= \langle \Delta C^{AB}, b_{\text{obs}}^{AB} \rangle - \underbrace{\langle L^* \lambda, b_S \rangle - B_1 \langle \lambda, b_S \rangle}_{(1)} \\
 &\quad - \langle \lambda, b_{\partial \tilde{t}^2 S} \rangle - \langle \nabla \lambda, b_{C=S} \rangle - B_2 \langle b_C, \lambda \rangle \\
 &\quad + \langle \lambda, b_f \rangle \\
 b_{\text{obs}}^{AB} &= f^+ b_S \\
 x &= \langle \Delta C^{AB} f^+, b_S \rangle - \langle L^* \lambda, b_S \rangle - \langle \lambda, b_{\partial \tilde{t}^2 S} \rangle - \langle \nabla \lambda, b_C \rangle \\
 &\quad + \langle \lambda, b_f \rangle \\
 &\quad + \langle -B_1 \langle \lambda, b_S \rangle - B_2 \langle \lambda, b_C \rangle
 \end{aligned}$$

3. The principle of noise tomography

$$bC_{obs}^{AB} = f^\dagger bS$$

$$\Rightarrow \delta\chi = \langle \Delta C^{AB} f^\dagger, bS \rangle - \langle L^\dagger \lambda, bS \rangle - \langle \lambda, b\rho \tilde{\partial}_b^2 S \rangle - \langle \langle \lambda \rangle, bC : vS \rangle$$

$$+ \langle \lambda, bf \rangle$$

$$+ \langle -B_1 \langle \lambda, bS \rangle - B_2 \langle \lambda, bC \rangle$$

$$\left. \begin{aligned} \text{if } L^\dagger \lambda &= \Delta C^{AB} f^\dagger \\ B_1 \langle \lambda, bS \rangle + B_2 \langle \lambda, bC \rangle &= 0 \end{aligned} \right\}$$

$$\text{Then } b\chi = -\langle \lambda, b\rho \tilde{\partial}_b^2 S \rangle - \langle \langle \lambda \rangle, bC : vS \rangle + \langle \lambda, bf \rangle$$