

Fast Marching Method and Some Latest Results

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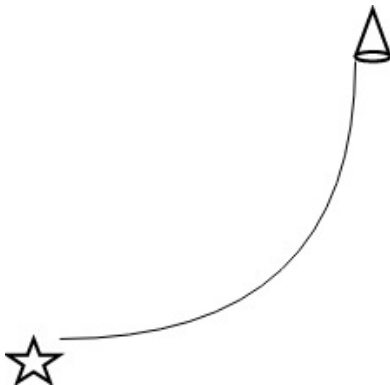
Outline

- 1 Level Set Concept
- 2 Fast Marching Methods
- 3 Multistencils Fast Marching Methods
- 4 Upwind Condition
- 5 Convergence
- 6 Other Related Numerical Methods

Level Set VS Particle Movement

The closed curve Γ represented by an auxiliary function ϕ .

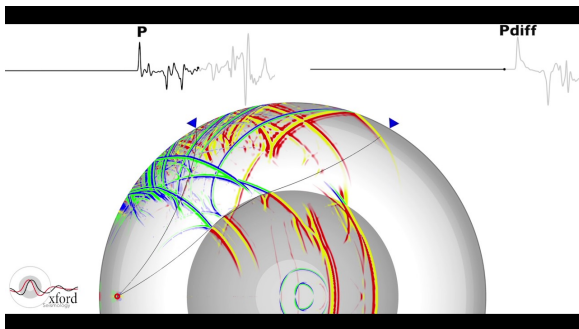
$$\Gamma = \{(x, y) \in \Omega | \phi(x, y) = 0\},$$



Level Set VS Particle Movement

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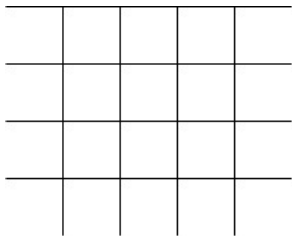
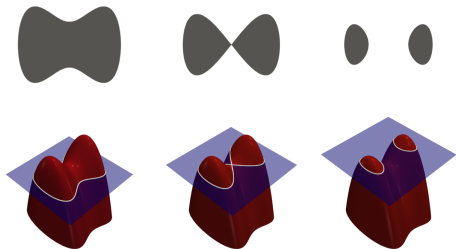
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Level Set VS Particle Movement

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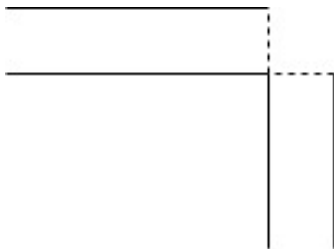
$$\Gamma = \{(x, y) \in \Omega | \phi(x, y) = 0\},$$



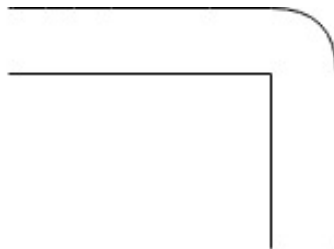
Level Set VS Particle Movement

Particle Movement: source and station \rightarrow ray path.

Level Set: source propagation timetable \rightarrow many paths (stations).



a



b

Level Set VS Fast Marching.(J.A.Sethian)

Initial Value Formulation.

$$\begin{aligned}\phi_t + \bar{F}|\nabla\phi| &= 0, \\ \text{Front} = \Gamma(t) &= \{(x, y) | \phi(x, y, t) = 0\},\end{aligned}$$

Applies for arbitrary \bar{F} .

Boundary Value Formulation.

$$\begin{aligned}|\nabla T|\bar{F} &= 1, \\ \text{Front} = \Gamma(t) &= \{(x, y) | T(x, y) = t\},\end{aligned}$$

Requires $\bar{F} > 0$.

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Nodes Tags

Know. The computed travel time at x is settled.

Narrow band. The computed travel time at x may be changed later.

Far. The travel time at x is set to be ∞ (not computed).

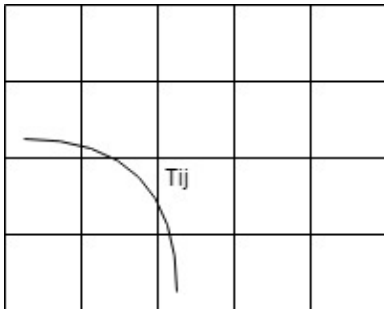
Loop

- 1 Among all narrow-band points, extract the point with minimum arrival time and change its tag to known.
- 2 Find its nearest neighbors that are either far or narrow band.
- 3 Update their arrival times.
- 4 Go back to one.

Animation

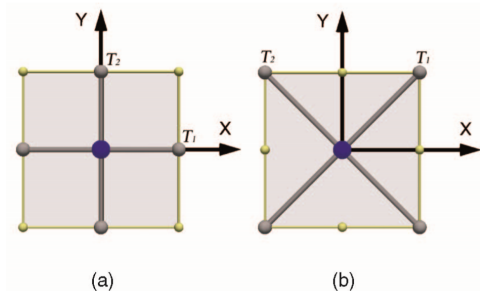
How to update the arrival time?

$$\max\left(\frac{T_{i,j} - T_{i-1,j}}{\delta x}, \frac{T_{i,j} - T_{i+1,j}}{\delta x}, 0\right)^2 + \max\left(\frac{T_{i,j} - T_{i,j-1}}{\delta y}, \frac{T_{i,j} - T_{i,j+1}}{\delta y}, 0\right)^2 = \frac{1}{F_{ij}^2}$$



Outline

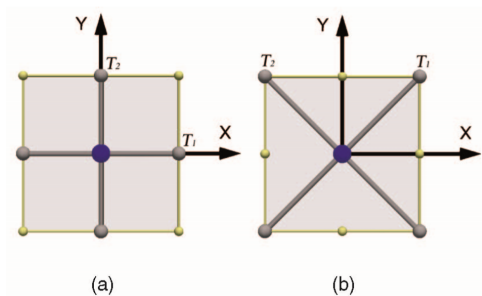
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Multistencils Fast Marching Method.(M. Sabry Hassouna and Aly A. Farag)

For the second stencil:

$$\frac{1}{F_{ij}^2} = \max\left(\frac{T_{i,j} - T_{i-1,j-1}}{\sqrt{2}h}, \frac{T_{i,j} - T_{i+1,j+1}}{\sqrt{2}h}, 0\right)^2 + \max\left(\frac{T_{i,j} - T_{i+1,j-1}}{\sqrt{2}h}, \frac{T_{i,j} - T_{i-1,j+1}}{\sqrt{2}h}, 0\right)^2$$



Multistencils Fast Marching Method.(M. Sabry Hassouna and Aly A. Farag)

For the second stencil, if we have an arbitrary angle θ between the directional vectors:

$$DT_1^2 - 2DT_1DT_2 \cos \theta + DT_2^2 = \frac{\sin^2 \theta}{F^2(X_0)}$$

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Upwind Condition for One Stencil

We use X_0, X_1, X_2 to denote the nodes and T_0, T_1, T_2 to denote its arrival time. Suppose we know T_1, T_2 at X_1, X_2 and we want to get T_0 at the upwind nodes X_0 . The discrete derivatives at direction $N = [N_1 N_2]$ are defined as

$$D_i T = \frac{T_0 - T_i}{|X_0 - X_i|}, i = 1, 2.$$

where

$$\begin{aligned} N_1 &= \frac{X_0 - X_1}{|X_0 - X_1|} \\ N_2 &= \frac{X_0 - X_2}{|X_0 - X_2|} \end{aligned}$$

Then

$$DT = N \nabla T$$

Upwind Condition for One Stencil

Thus we have

$$DT^T(NN^T)^{-1}DT = \frac{1}{F^2}.$$

Meanwhile the upwind condition requires

$$(NN^T)^{-1}DT > 0.$$

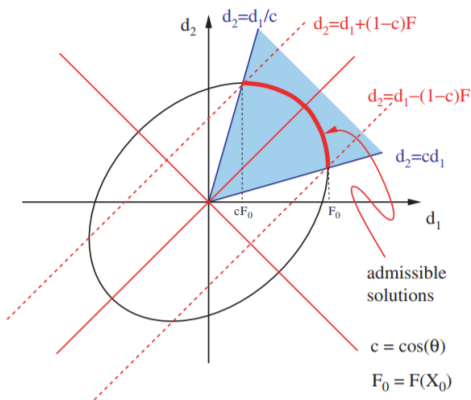
From

$$DT_1^2 - 2DT_1DT_2 \cos \theta + DT_2^2 = \frac{\sin^2 \theta}{F^2(X_0)}$$

Upwind Condition for One Stencil

$$DT_1^2 - 2DT_1DT_2 \cos \theta + DT_2^2 = \frac{\sin^2 \theta}{F^2(X_0)}$$

$$DT_1 - \cos \theta DT_2 > 0, DT_2 - \cos \theta DT_1 > 0$$

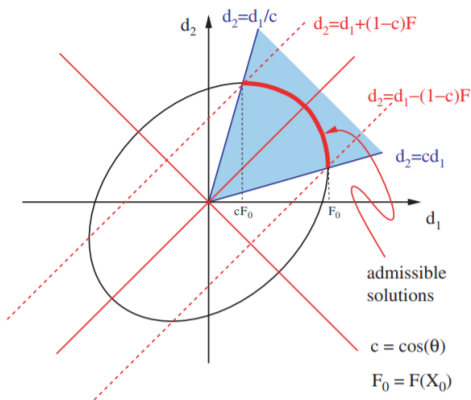


Upwind Condition for One Stencil

which give us

$$|DT_1 - DT_2| = \frac{1 - \cos \theta}{F(X_0)}$$

here if $c = \cos \theta$



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Convergence

Define

$$p = \frac{\log(L_{k,i}/L_{k,i+1})}{\log(h_i/h_{i+1})}$$

with grid size h_i and error $L_{k,i}$ (grid set i , norm k). where

$$L_1 = \frac{1}{n} \sum_{i=1}^n |T - T_a|, L_2 = \frac{1}{n} \sum_{i=1}^n |T - T_a|^2, L_\infty = \max(|T - T_a|)$$

Convergence of the Proposed Methods in 2D Space

h	MSFM ₁						MSFM ₂					
	L_1	p	L_2	p	L_∞	p	L_1	p	L_2	p	L_∞	p
1/21	3.85e-03	-	5.07e-03	-	1.07e-02	-	1.76e-03	-	2.18E-03	-	3.51e-03	-
1/41	2.70e-03	0.51	3.42e-03	0.56	7.22e-03	0.57	6.32e-04	1.48	7.28e-04	1.58	1.39e-03	1.33
1/81	1.57e-03	0.78	1.97e-03	0.79	4.21e-03	0.78	2.03e-04	1.63	2.34e-04	1.64	4.95e-04	1.5
1/161	8.47e-04	0.89	1.06e-03	0.89	2.30e-03	0.87	5.78e-05	1.81	6.67e-05	1.81	1.44e-04	1.78
1/321	4.40e-04	0.94	5.52e-04	0.95	1.21e-03	0.92	1.54e-05	1.90	1.78e-05	1.90	3.85e-05	1.90
1/641	2.24e-04	0.97	2.81e-04	0.97	6.30e-04	0.95	4.02e-06	1.95	4.62e-06	1.95	9.94e-06	1.96
1/1281	1.13e-04	0.99	1.42e-04	0.99	3.22e-04	0.97	1.02e-06	1.97	1.17e-06	1.97	2.52e-06	1.98
1/2561	5.56e-05	1.02	6.67e-05	1.09	1.61e-04	1.00	2.59e-07	1.98	2.98e-07	1.98	6.36e-07	1.99

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Fast Sweeping Methods

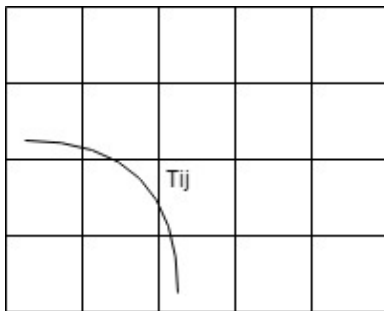
We sweep the whole domain with four alternating orderings repeatedly:

$$1 \quad i = 1 : I, j = 1 : J,$$

$$2 \quad i = I : 1, j = 1 : J,$$

$$3 \quad i = I : 1, j = J : 1,$$

$$4 \quad i = 1 : I, j = J : 1.$$



Improved Fast Marching Methods

Accuracy.

- 1 The Higher Accuracy Fast Marching Method (HAFMM) : second or higher order difference scheme.
- 2 The Shifted Grid Fast Marching (SGFM) Method.

Time efficiency.

- 3 The Group Marching Method (GMM).
- 4 Untidy Fast Marching Method (UFMM).

Thank you!