

Title: Double-difference adjoint seismic tomography

**Authors: Yanhua O. Yuan, Frederik J. Simons and
Jeroen Tromp**

Cross-correlation travelttime difference

$$\chi_{cc} = \frac{1}{2} \sum_{r=1}^N \left[T_r(m) - T_r^{obs} \right]^2$$

$$\Delta t_r = \arg \max_{\tau} \int_0^T w_r s_i(x_r, t + \tau) d_i(x_r, t) dt$$

$$\begin{aligned}
C(\tau) &= \int_0^T w_r s_i(t+\tau) d_i(t) dt = \int_0^T w_r s_i(t+\tau) (s_i(t) + \delta s_i(t)) dt \\
&= \int_0^T w_r \left(s_i(t) + \frac{\partial s_i(t)}{\partial t} \tau + \frac{\partial^2 s_i(t)}{\partial t^2} \tau^2 \right) (s_i(t) + \delta s_i(t)) dt \\
&= \int_0^T w_r s_i(t) \left(s_i(t) + \frac{\partial s_i(t)}{\partial t} \tau + \frac{1}{2} \frac{\partial^2 s_i(t)}{\partial t^2} \tau^2 \right) dt + \\
&\quad \int_0^T w_r \delta s_i(t) \left(s_i(t) + \frac{\partial s_i(t)}{\partial t} \tau \right) dt \\
\frac{\partial C(\tau)}{\partial \tau} &= \int_0^T w_r s_i(t) \left(\frac{\partial s_i(t)}{\partial t} + \frac{\partial^2 s_i(t)}{\partial t^2} \tau \right) dt + \int_0^T w_r \delta s_i(t) \frac{\partial s_i(t)}{\partial t} dt
\end{aligned}$$

$$\frac{\partial \mathcal{C}(\Delta t_i)}{\partial \tau} = \int_0^T w_r s_i(t) \left(\frac{\partial s_i(t)}{\partial t} + \frac{\partial^2 s_i(t)}{\partial t^2} \Delta t_i \right) dt + \int_0^T w_r \delta s_i(t) \frac{\partial s_i(t)}{\partial t} dt = 0$$

$$\Delta t_r = - \frac{\int_0^T w_r \delta s_i(t) \frac{\partial s_i(t)}{\partial t} dt}{N_r}$$

$$N_r = \int_0^T w_r s_i(t) \frac{\partial^2 s_i(t)}{\partial t^2} dt$$

$$\delta \chi_{cc} = \sum_{r=1}^N \left[T_r(m) - T_r^{obs} \right] \delta T_r(m)$$

$$\delta\chi_{cc} = -\sum_{r=1}^N \left[T_r(m) - T_r^{obs} \right] \frac{\int_0^T w_r \delta s_i(x_r, t) \frac{\partial s_i(x_r, t)}{\partial t} dt}{N_r(x_r)}$$

$$(\rho, \mathbf{c}) \rightarrow (\rho + \delta\rho, \mathbf{c} + \delta\mathbf{c})$$

$$\mathbf{s} \rightarrow \mathbf{s} + \delta\mathbf{s}$$

$$\rho \partial_t^2 \mathbf{s} - \nabla \cdot \mathbf{c} : \nabla \mathbf{s} = \mathbf{f}$$

$$(\rho + \delta\rho) \partial_t^2 (\mathbf{s} + \delta\mathbf{s}) - \nabla \cdot (\mathbf{c} + \delta\mathbf{c}) : \nabla (\mathbf{s} + \delta\mathbf{s}) = \mathbf{f}$$

$$\rho \partial_t^2 \delta \mathbf{s} - \nabla \cdot \mathbf{c} : \nabla \delta \mathbf{s} = -\delta \rho \partial_t^2 \mathbf{s} + \nabla \cdot \delta \mathbf{c} : \nabla \mathbf{s}$$

$$\delta s_i(\mathbf{x}, t) = - \int_0^t \int_V [\delta \rho(\mathbf{x}') G_{ij}(\mathbf{x}, \mathbf{x}'; t - t') \partial_{t'}^2 s_j(\mathbf{x}', t') \cdot$$

$$+ \delta c_{jklm}(\mathbf{x}') \partial'_k G_{ij}(\mathbf{x}, \mathbf{x}'; t - t') \partial'_l s_m(\mathbf{x}', t')] d^3 \mathbf{x}' dt'$$

$$\begin{aligned}
\delta\chi_{cc} &= -\sum_{r=1}^N \frac{[T_r(m) - T_r^{obs}]}{N_r(x_r)} \int_0^T w_r \delta S_i(x_r, t) \frac{\partial S_i(x_r, t)}{\partial t} dt \\
&= \sum_{r=1}^N \frac{[T_r(m) - T_r^{obs}]}{N_r(x_r)} \int_0^T w_r \frac{\partial S_i(x_r, t)}{\partial t} \int_0^t \int_V [\delta\rho(x') G_{ij}(x_r, x'; t - t') \\
&\quad \partial_{t'}^2 S_j(x', t') + \delta c_{iklm}(x') \partial_k G_{ij}(x_r, x'; t - t') \partial_k S_m(x', t')] d^3 x' dt' dt
\end{aligned}$$

Let us define the field

$$\Phi_k(x', t') = \sum_{r=1}^N \frac{[T_r(m) - T_r^{obs}]}{N_r(x_r)} \int_{t'}^T w_r G_{ik}(x_r, x'; t - t') \frac{\partial S_i(x_r, t)}{\partial t} dt$$

$$G_{ik}(x_r, x'; t - t') = G_{ki}(x', x_r; t - t')$$

$$\Phi_k(x', t') = \sum_{r=1}^N \frac{[T_r(m) - T_r^{obs}]}{N_r(x_r)} \int_{t'}^T w_r G_{ki}(x', x_r; t - t') \frac{\partial s_i(x_r, t)}{\partial t} dt$$

$$t \rightarrow T - t$$

$$\Phi_k(x', t') = - \sum_{r=1}^N \frac{[T_r(m) - T_r^{obs}]}{N_r(x_r)} \int_{T-t'}^0 w_r G_{ki}(x', x_r; T - t - t') \frac{\partial s_i(x_r, T - t)}{\partial t} dt$$

$$= \sum_{r=1}^N \frac{[T_r(m) - T_r^{obs}]}{N_r(x_r)} \int_0^{T-t'} w_r G_{ki}(x', x_r; T - t - t') \frac{\partial s_i(x_r, T - t)}{\partial t} dt$$

$$f_i^\dagger(x, t) = \sum_{r=1}^N \frac{[T_r(m) - T_r^{obs}]}{N_r(x_r)} w_r \frac{\partial s_i(x_r, T - t)}{\partial t} \delta(x - x_r)$$

$$\Phi_k(x', t') = \int_0^{T-t'} \int_V w_r G_{ki}(x', x_r; T - t - t') f_i^\dagger(x, t) d^3x dt$$

$$t' \rightarrow T - t'$$

$$s_k^\dagger(x', t') = \int_0^{t'} \int_V G_{ki}^\dagger(x', x, t' - t) f_i^\dagger(x, t) d^3x dt$$

$$s_k^\dagger(x', T - t') = \Phi_k(x', t')$$

$$\delta\chi_{cc} = \int_V \int_0^T \left[s^\dagger(x', T - t') \cdot \partial_t^2 s \delta\rho(x') dt + \nabla s^\dagger(x', T - t') : \delta c : \nabla s \right] dt d^3x$$

$$K_\rho = \int_0^T s^\dagger(x', T - t') \cdot \partial_t^2 s dt$$

$$\mathbf{K}_c = \int_0^T \nabla s^\dagger(x', T - t') \nabla s dt \quad \delta\chi_{cc} = \int_V K_\rho \delta\rho + \mathbf{K}_c :: \delta c d^3x$$

Double-difference cross-correlation

$$\Delta t_{ij}^{\text{syn}} = \operatorname{argmax}_{\tau} \int_0^T s_i(t + \tau) s_j(t) dt, \quad \text{and we define} \quad \Gamma_{ij}(\tau) = \int_0^T s_i(t + \tau) s_j(t) dt,$$

$$\Delta t_{ij}^{\text{obs}} = \operatorname{argmax}_{\tau} \int_0^T d_i(t + \tau) d_j(t) dt, \quad \text{and we define} \quad \Lambda_{ij}(\tau) = \int_0^T d_i(t + \tau) d_j(t) dt.$$

$$\Delta \Delta t_{ij} = \Delta t_{ij}^{\text{syn}} - \Delta t_{ij}^{\text{obs}}.$$

$$\chi_{\text{cc}}^{\text{dd}} = \frac{1}{2} \sum_i \sum_{j>i} [\Delta \Delta t_{ij}]^2.$$

$$\Delta t_{ji}^{\text{syn}} = -\Delta t_{ij}^{\text{syn}} \quad \text{and} \quad \Delta t_{ji}^{\text{obs}} = -\Delta t_{ij}^{\text{obs}}$$

$$\tilde{s}_i(t) = s_i(t) + \delta s_i(t) \quad \tilde{s}_j(t) = s_j(t) + \delta s_j(t).$$

$$\tilde{\Gamma}_{ij}(\tau) = \int_0^T \tilde{s}_i(t + \tau) \tilde{s}_j(t) dt \approx \Gamma_{ij}(\tau) + \int_0^T \delta s_i(t + \tau) s_j(t) dt +$$

$$\int_0^T s_i(t + \tau) \delta s_j(t) dt = \Gamma_{ij}(\tau) + \delta \Gamma_i(\tau) + \delta \Gamma_j(\tau).$$

$$\begin{aligned} \tilde{\Gamma}_{ij}(\Delta t_{ij}^{\text{syn}} + \delta\tau) &= \Gamma_{ij}(\Delta t_{ij}^{\text{syn}}) + \delta\Gamma_i(\Delta t_{ij}^{\text{syn}}) + \delta\Gamma_j(\Delta t_{ij}^{\text{syn}}) \\ &+ \delta\tau \partial_\tau \delta\Gamma_i(\Delta t_{ij}^{\text{syn}}) + \delta\tau \partial_\tau \delta\Gamma_j(\Delta t_{ij}^{\text{syn}}) + \frac{1}{2}\delta\tau^2 \partial_\tau^2 \Gamma_{ij}(\Delta t_{ij}^{\text{syn}}) \end{aligned}$$

$$\partial_{\delta\tau} \tilde{\Gamma}_{ij}(\Delta t_{ij}^{\text{syn}} + \delta\tau) = \delta\tau \partial_\tau^2 \Gamma_{ij}(\Delta t_{ij}^{\text{syn}}) + \partial_\tau \delta\Gamma_i(\Delta t_{ij}^{\text{syn}}) + \partial_\tau \delta\Gamma_j(\Delta t_{ij}^{\text{syn}}) = 0.$$

$$\begin{aligned} \delta\Delta t_{ij}^{\text{syn}} &= -\frac{\partial_\tau \delta\Gamma_i(\Delta t_{ij}^{\text{syn}}) + \partial_\tau \delta\Gamma_j(\Delta t_{ij}^{\text{syn}})}{\partial_\tau^2 \Gamma_{ij}(\Delta t_{ij}^{\text{syn}})} \\ &= \frac{\int_0^T \partial_t s_j(t - \Delta t_{ij}^{\text{syn}}) \delta s_i(t) dt - \int_0^T \partial_t s_i(t + \Delta t_{ij}^{\text{syn}}) \delta s_j(t) dt}{\int_0^T \partial_t^2 s_i(t + \Delta t_{ij}^{\text{syn}}) s_j(t) dt} \end{aligned}$$

$$N_{ij} = \int_0^T \partial_t^2 s_i(t + \Delta t_{ij}^{\text{syn}}) s_j(t) dt,$$

$$\delta \chi_{\text{cc}}^{\text{dd}} = \int_0^T \left\{ \sum_i \left[\sum_{j>i} \frac{\Delta \Delta t_{ij}}{N_{ij}} \partial_t s_j(t - \Delta t_{ij}^{\text{syn}}) \right] \delta s_i(t) \right. \\ \left. - \sum_j \left[\sum_{i<j} \frac{\Delta \Delta t_{ij}}{N_{ij}} \partial_t s_i(t + \Delta t_{ij}^{\text{syn}}) \right] \delta s_j(t) \right\} dt$$

$$f_i^\dagger(\mathbf{x}, t) = + \sum_{j>i} \frac{\Delta \Delta t_{ij}}{N_{ij}} \partial_t s_j (T - [t - \Delta t_{ij}^{\text{syn}}]) \delta(\mathbf{x} - \mathbf{x}_i),$$

$$f_j^\dagger(\mathbf{x}, t) = - \sum_{i<j} \frac{\Delta \Delta t_{ij}}{N_{ij}} \partial_t s_i (T - [t + \Delta t_{ij}^{\text{syn}}]) \delta(\mathbf{x} - \mathbf{x}_j).$$

$$\delta \chi_{\text{cc}}^{\text{dd}} = \int_{\oplus} K_m^{\text{dd}}(\mathbf{x}) m(\mathbf{x}) d^3 \mathbf{x},$$

$$\begin{aligned}
& \operatorname{argmax}_{\tau} \int_0^T [\alpha s_i(t + \tau)][\alpha s_j(t)] dt = \operatorname{argmax}_{\tau} \left[\alpha^2 \int_0^T s_i(t + \tau) s_j(t) dt \right] \\
& = \operatorname{argmax}_{\tau} \int_0^T s_i(t + \tau) s_j(t) dt = \Delta t_{ij}^{\text{syn}} \\
& \dots \dots \dots \\
& \operatorname{argmax}_{\tau} \int_0^T s_i(t + \tau - \tau_0) s_j(t - \tau_0) dt = \operatorname{argmax}_{\tau} \int_0^T s_i(t + \tau) s_j(t) dt = \Delta t_{ij}^{\text{syn}}
\end{aligned}$$





