

Mesh Design for Fast Marching Method

Qi Yingyu

Supervisor: Prof. Tong Ping

Nanyang Technological University, Singapore

Outline

1 Background

2 Tetrahedral mesh

3 Numerical results

4 Conclusion

Design an Algorithm — Fast marching

Purpose

- 1 Time complexity.
 - 2 Convergency.
 - 3 Accuracy.
-
- 1 Equation: simplification, factorization,
 - 2 Iteration scheme: Gauss-Seidel, Gudunov,
 - 3 Difference scheme: Higher orders, switch, staggered grid, multi-stencils,
 - 4 Mesh: ?

Mesh Design

- 1 2D: Square → Triangle.
- 2 3D: Cuboid → Tetrahedral.

Achievements

- 1 Nonlinear phase.
- 2 reflection and refraction waves.
- 3 complicated subsurface geometry.
- 4 increasing cell volumes.

Outline

1 Background

2 Tetrahedral mesh

3 Numerical results

4 Conclusion

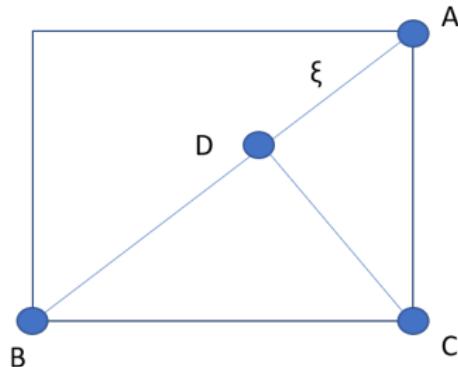


Figure 1: from rectangular mesh to triangle mesh

from rectangular mesh to triangle mesh

$$t(D) = t(A) + \xi(t(B) - t(A))$$

$$t(C) = t(D) + s|\vec{DC}|$$

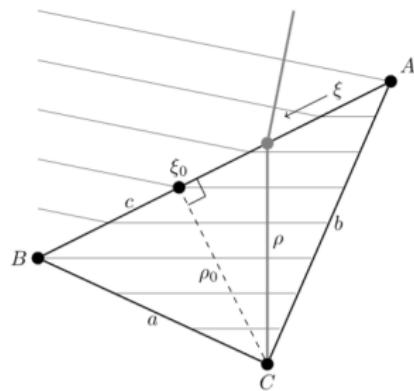


Figure 2: triangle mesh

triangle mesh

$$t_C(\xi) = t_A + \xi(t_B - t_A) + s\sqrt{c^2(\xi - \xi_0)^2 + \rho_0^2}$$

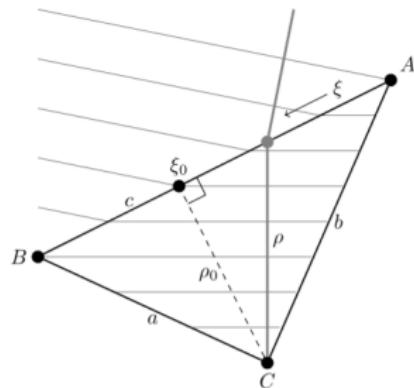
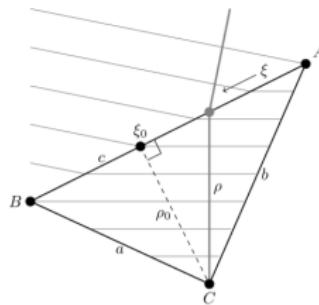


Figure 3: triangle mesh

triangle mesh

$$\frac{dt_C(\xi)}{d\xi} = (t_B - t_A) + \frac{sc^2(\xi - \xi_0)}{\sqrt{c^2(\xi - \xi_0)^2 + \rho_0^2}} = 0$$



triangle mesh

$$(\xi - \xi_0) = \pm \frac{\rho_0(t_B - t_A)}{c \sqrt{s^2 c^2 - (t_B - t_A)^2}}$$

$$\rho = \mp \frac{s c \rho_0}{\sqrt{s^2 c^2 - (t_B - t_A)^2}}$$

$$t_C = t_A + (t_B - t_A)\xi_0 + \frac{\rho_0 \sqrt{s^2 c^2 - (t_B - t_A)^2}}{c}$$

$$t = \min\{t_C, t_A + s|\vec{AC}|, t_B + s|\vec{BC}|\}$$

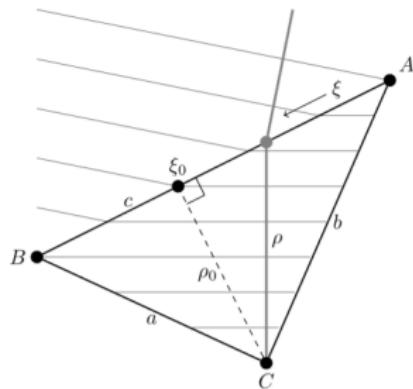


Figure 4: triangle mesh

upwind condition

$$\begin{aligned} 0 &< \xi < 1 \\ w^2 &= s^2 c^2 - (t_B - t_A)^2 \geq 0 \end{aligned}$$

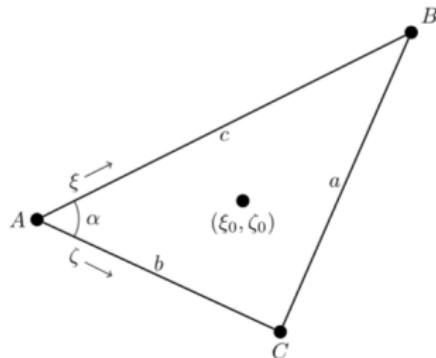


Figure 5: tetrahedral mesh

objective function

$$\begin{aligned}
 t_D(\xi, \zeta) &= t_A + (t_B - t_A)\xi + (t_C - t_A)\zeta + s\rho \\
 \rho^2 &= |(\xi - \xi_0)\vec{c} + (\zeta - \zeta_0)\vec{b}|^2 + \rho_0^2 \\
 &= c^2(\xi - \xi_0)^2 + b^2(\zeta - \zeta_0)^2 + 2bc \cos \alpha (\xi - \xi_0)(\zeta - \zeta_0) + \rho_0^2
 \end{aligned}$$

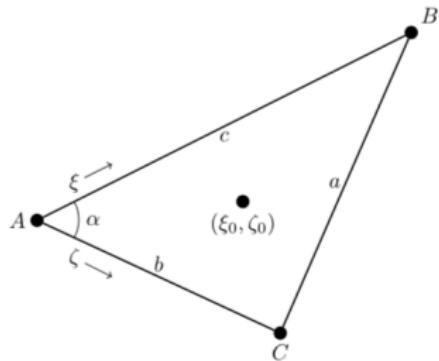


Figure 6: tetrahedral mesh

optimized solution

$$\begin{aligned}\frac{\partial t_D}{\partial \xi} &= (t_B - t_A) + \frac{s}{\rho}(c^2(\xi - \xi_0) + bc \cos \alpha(\zeta - \zeta_0)) = 0 \\ \frac{\partial t_D}{\partial \zeta} &= (t_C - t_A) + \frac{s}{\rho}(b^2(\zeta - \zeta_0) + bc \cos \alpha(\xi - \xi_0)) = 0\end{aligned}$$

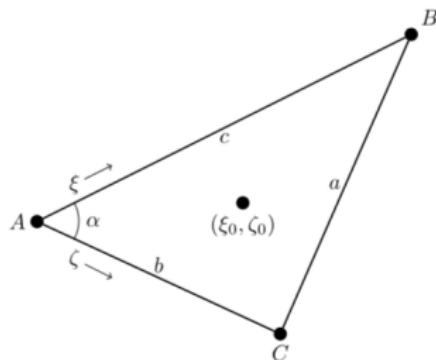


Figure 7: tetrahedral mesh

optimized solution

$$(\xi - \xi_0) = \frac{(t_B - t_A)b^2 - (t_C - t_A)bc \cos \alpha}{(t_C - t_A)c^2 - (t_B - t_A)bc \cos \alpha} (\zeta - \zeta_0)$$

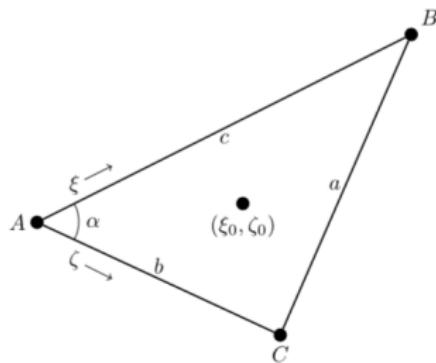


Figure 8: tetrahedral mesh

optimized solution

$$\begin{aligned}
 (\xi - \xi_0) &= -\frac{|(t_B - t_A)b^2 - (t_C - t_A)bc \cos \alpha| \rho_0}{\tilde{w}bc \sin \alpha} \\
 (\zeta - \zeta_0) &= -\frac{|(t_C - t_A)c^2 - (t_B - t_A)bc \cos \alpha| \rho_0}{\tilde{w}bc \sin \alpha} \\
 \tilde{w}^2 &= s^2 b^2 c^2 \sin^2 \alpha - (t_B - t_A)^2 b^2 - (t_C - t_A)^2 c^2 \\
 &\quad + 2(t_B - t_A)(t_C - t_A)bc \cos \alpha
 \end{aligned}$$

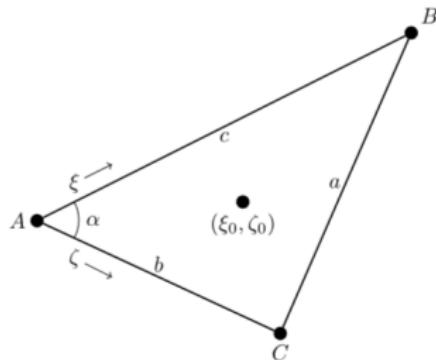


Figure 9: tetrahedral mesh

optimized solution

$$\begin{aligned}
 t_D &= t_A + (t_B - t_A)\xi_0 + (t_C - t_A)\zeta_0 + \frac{\tilde{w}\rho_0}{bc \sin \alpha} \\
 t &= \min\{t_D, t_A + s|\vec{AD}|, t_B + s|\vec{BD}|, t_C + s|\vec{CD}|\}
 \end{aligned}$$

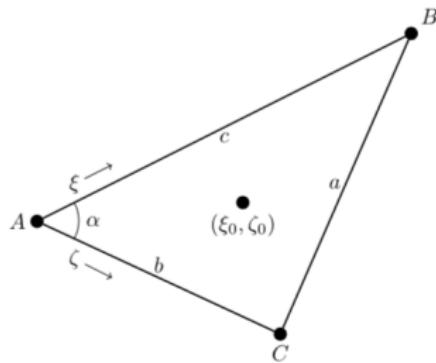


Figure 10: tetrahedral mesh

upwind condition

$$0 < \xi < 1$$

$$0 < \zeta < 1$$

$$0 < \xi + \zeta < 1$$

$$\begin{aligned} \tilde{w}^2 &= s^2 b^2 c^2 \sin^2 \alpha - (t_B - t_A)^2 b^2 - (t_C - t_A)^2 c^2 \\ &+ 2(t_B - t_A)(t_C - t_A)bc \cos \alpha \geq 0 \end{aligned}$$

Outline

- 1 Background
- 2 Tetrahedral mesh
- 3 Numerical results
- 4 Conclusion

Mesh cell

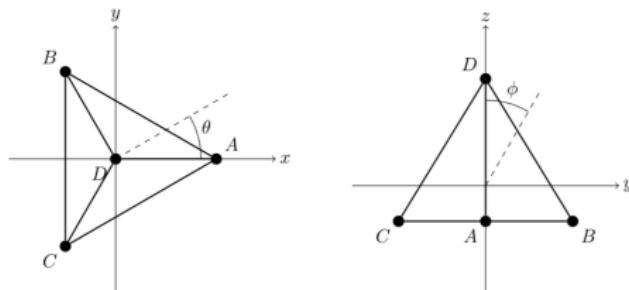


Figure 6. The tetrahedron defined in Table 1 as viewed from the $+z$ (left) and $+x$ (right) directions. We define an azimuthal angle θ and a polar angle ϕ as illustrated.

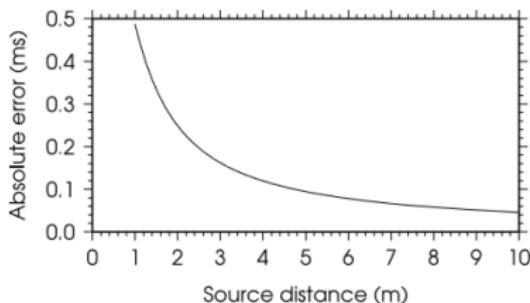


Figure 7. Absolute error versus source distance for the FMM local update through the tetrahedral cell in Table 1 and Fig. 6 with $\theta = 0$ and $\phi = 180$.

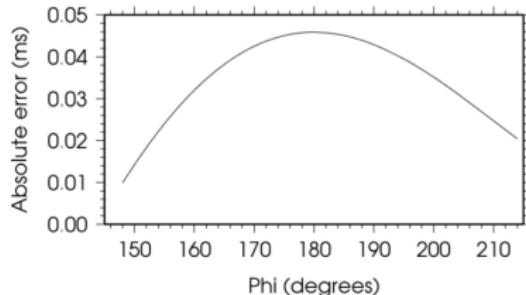


Figure 8. Absolute error versus ϕ for the FMM local update through the tetrahedral cell in Table 1 and Fig. 6 with $\theta = 0$ and source distance 10.0 m.

CPU time

Table 2. Maximum tetrahedral volumes supplied to TetGen, the resulting number of grid nodes and tetrahedral cells used to discretize a cube of 100 m dimensions, and the CPU computation time for the FMM solutions on those grids using a 2.53 GHz dual core machine.

Volume (m ³)	Nodes	Cells	Time (s)
1	307692	1889668	21.76
8	40441	236485	2.56
64	5627	30028	0.30
216	1803	8781	0.08

Table 3. Histogram information for obtuse dihedral angles contained in the meshes created by TetGen (see Table 2 for more information). Here we show percentages of cells with maximum dihedral angles within the indicated bins.

Volume (m ³)	(90,100] ^o	(100,110] ^o	(110,120] ^o	(120,170] ^o
1	27 per cent	19 per cent	12 per cent	19 per cent
8	27 per cent	19 per cent	13 per cent	19 per cent
64	26 per cent	19 per cent	13 per cent	19 per cent
216	27 per cent	19 per cent	13 per cent	20 per cent

Mesh-Error

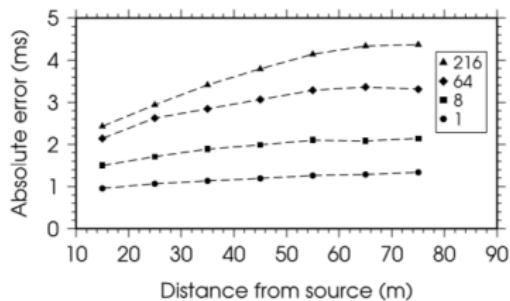


Figure 9. Absolute error versus nodal distance from source for four different discretizations. The distances are binned every 10 m and the maximum error is plotted for each bin (at the locations of the bin centres on the horizontal axis). The legend indicates maximum cell volumes in cubic metres.

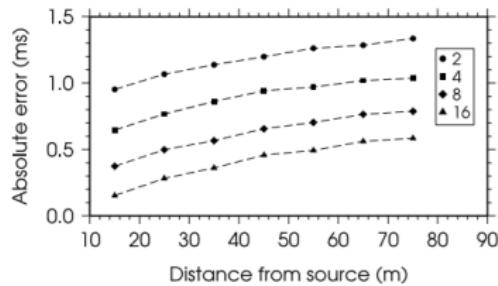


Figure 10. Absolute error versus nodal distance from source for four different initialization radii. The distances are binned every 10 m and the maximum error is plotted for each bin. The legend indicates initialization radii in metres.

Outline

1 Background

2 Tetrahedral mesh

3 Numerical results

4 Conclusion

Conclusion

- extend the Fast Marching Method for use on unstructured 3-D tetrahedral grids.
- first-order accuracy or worse if obtuse dihedral angles exist.

Thank you!