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014

A VISCOSITY SOLUTIONS APPROACH TO SHAPE-FROM-SHADING*

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Abstract. The problem of recovering a Lambertian surface from a single two-dimensional image may be written as a first-order nonlinear equation which presents the disadvantage of having several continuous and even smooth solutions. A new approach based on Hamilton–Jacobi–Bellman equations and viscosity solutions theories enables one to study non-uniqueness phenomenon and thus to characterize the surface among the various solutions.

A consistent and monotone scheme approximating the surface is constructed thanks to the dynamic programming principle, and numerical results are presented.

Key words. viscosity solutions, Hamilton–Jacobi equations, dynamic programming approximation scheme, shape-from-shading

AMS(MOS) subject classifications. 65M15, 65M10

1. The shape-from-shading problem.

1.1. Introduction. The purpose of this work is the construction of an algorithm computing the solution of some first-order Hamilton–Jacobi equation whose origins are in image analysis and, more precisely, in shape-from-shading problems.

The shape of a surface is related to the image brightness by the Horn image irradiance equation (see Horn [14, Chap. 10]).

Given the brightness value $I(x, y)$ at each pixel and the reflectance map $R(n)$ which specifies the reflectance of a surface as a function of its orientation, the image irradiance equation may be written

$$R(n(x, y)) = I(x, y).$$

The reflectance map depends on the reflectance properties of the surface and the distribution of light sources. The image irradiance equation may be considered as a nonlinear first-order equation whose unknown variable is the elevation of the shape.

Horn first solved this equation by a global method under some assumptions on surface smoothness which provided the additional information necessary to characterize the shape (see [14, Chap. 11]). He then developed another formulation of the problem that consists in minimizing a certain composite function of either the orientation or the elevation of the surface involving a regularization term. This method yields more efficient numerical schemes.

In the case of a single distant light source, Pentland [18]–[20] recently proposed an algorithm that estimates shape from local variations in the image intensity by computing the solution in the Fourier domain. The usual assumption on surface smoothness is no longer necessary and the reflectance map is approximated by a linear function of the partial derivatives. The algorithm gives a good estimate of the surface for high frequencies and may be united with a stereo treatment for low ones.

We propose here a new method that consists in solving directly the image irradiance equation, which is a PDE with boundary data. First, we want to restrict our study to continuous shapes. This requires the use of viscosity solutions whose theory has been developed by Crandall and Lions [10] and many others.

The next part of this paper is dedicated to the statement of a shape-from-shading model and its presentation in terms of the Hamilton–Jacobi equation.

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In the following section we recall some notions of viscosity solutions and present a uniqueness result which will allow us to characterize the shape among the solutions. Moreover we give conditions for the existence of both continuous and smooth solutions.

In § 3 we state the link between viscosity solutions and optimal control theories via the dynamic programming principle. It yields a stable, consistent and monotone numerical scheme which converges uniformly toward the viscosity solution. We solve the discrete equations of this stationary scheme thanks to an optimal algorithm based on its monotonicity.

Finally, we present some numerical results and propose some related shape-from-shading problems which could be solved in a similar way.

1.2. A shape-from-shading model. In this work we concentrate on the recovery of a Lambertian surface illuminated by a single distant light source. In this case, the reflectance map $R(n(x, y))$ has a well-known and simple form.

Let Ω be an open subset of $\mathbb{R}^2$, the scalar function u defined on Ω the elevation of the shape, and (α, β, γ) the unit vector in the illuminant direction where α, β, γ are nonnegative real numbers.

In the case of a Lambertian and not self-shadowing surface illuminated by a single distant light source, we have

$$I(x, y) = R(n(x, y)) = n \cdot (\alpha, \beta, \gamma)$$
$$= \frac{\alpha\, \partial u/\partial x + \beta\, \partial u/\partial y + \gamma}{\sqrt{1 + (\partial u/\partial x)^2 + (\partial u/\partial y)^2}}.$$

We assume that I is a positive, Lipschitz continuous function; the above equation may be rewritten as a first-order Hamilton-Jacobi equation:

$$I(x)\sqrt{1 + |\nabla u|^2} - (\alpha, \beta) \cdot \nabla u - \gamma = 0 \quad \forall x \in \Omega, \tag{1}$$

where the Hamiltonian of the problem,

$$H(x, p) = I(x)\sqrt{1 + |p|^2} - (\alpha, \beta) \cdot p - \gamma, \qquad x \in \Omega, \quad p \in \mathbb{R}^2,$$

is Lipschitz continuous with respect to its first argument and convex continuous with respect to its second one.

Obviously we need conditions on the boundary of the domain Ω for the Hamiltonian does not depend on u. We choose here a Dirichlet type boundary condition and more precisely we consider the following example

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

even if everything we do adapts trivially to arbitrary boundary data.

2. Hamilton-Jacobi equations and viscosity solutions.

2.1. Definitions. We first recall the definition of viscosity solutions. This notion, introduced first by Crandall and Lions in [10], is a notion of weak solutions of first-order nonlinear equations, called Hamilton-Jacobi equations.

We consider the following equation:

$$H(x, u(x), \nabla u(x)) = 0 \quad \text{in } \Omega, \tag{3}$$

where Ω is an open subset of $\mathbb{R}^n$ and H, the Hamiltonian, is a continuous scalar function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$.

$u \in \mathrm{BUC}(\Omega)$ (bounded, uniformly continuous on Ω) is a viscosity subsolution of (3) if for all $\varphi \in \mathbf{C}^1(\Omega)$, for all $x_0 \in \Omega$ local maximum of $u - \varphi$,

$$H(x_0, u(x_0), \nabla\varphi(x_0)) \leqq 0$$

holds, and $u \in \mathrm{BUC}\,(\Omega)$ is a viscosity supersolution of (3) if for all $\varphi \in \mathbf{C}^1(\Omega)$, for all $x_0 \in \Omega$ local minimum of $u-\varphi$,

$$H(x_0, u(x_0), \nabla\varphi(x_0)) \geqq 0$$

holds. Finally, u is a viscosity solution of (3) if u is a viscosity subsolution and supersolution of (3).

Remark 1. Let u be a viscosity solution of (3). If $u \in \mathbf{C}^1(\Omega)$, then u is a classical solution of (3) (i.e., (3) holds at each point) and if $u \in W^{1,\infty}(\Omega)$, the equation holds almost everywhere. Conversely, if $u \in \mathbf{C}^1(\Omega)$ solves (3) then u is a viscosity solution of (3).

2.2. A uniqueness result. We now present a stability result, in case the Hamiltonian does not depend on u, due to Ishii [15] and which can also be found with a different proof in Lions [16]. It states that the positive difference between a viscosity subsolution and a viscosity supersolution is inferior to its value on the boundary. We need this result in its uniqueness version in order to ensure the convergence of our algorithm towards the right solution.

THEOREM 1. *Let $u, v \in \mathrm{BUC}\,(\bar\Omega)$ be, respectively, viscosity sub- and supersolutions of the following equation*:

$$H(x, \nabla u(x)) = 0 \quad \textit{in } \Omega \textit{ bounded open subset of } \mathbb{R}^n, \tag{4}$$

where H is continuous on $\bar\Omega \times \mathbb{R}^n$ and convex in ∇u. In addition we assume that

$$\forall x, y \in \Omega, \forall p \in \mathbb{R}^n, \ |H(x,p) - H(y,p)| \leqq \omega(|x-y|(1+|p|))$$
$$\textit{with } \omega \textit{ continuous nondecreasing function such that } \omega(0) = 0$$

holds and that there exists a strict viscosity subsolution $\underline{u} \in \mathbf{C}^1(\Omega) \cap \mathbf{C}(\bar\Omega)$ of (4) (*i.e., such that $H(x, \nabla\underline{u}(x)) < 0$, for all $x \in \Omega$*).

Then we have

$$\sup_{\bar\Omega}\,(u-v)^+ \leqq \sup_{\partial\Omega}\,(u-v)^+.$$

Remark 2. Let $v \in \mathbf{C}(\bar\Omega)$ be defined by

$$u = \underline{u} + e^v.$$

If u is a viscosity solution of (4) then v is a viscosity solution of $H'(x, v, \nabla v) = 0$ in Ω, where

$$H'(x, t, p) = H(x, \nabla\underline{u} + p\,e^t)\,e^{-t}$$

is a nondecreasing function with respect to t. We may then use comparison theorems contained in Barles [2] or Lions [16] to find the same results, the partial derivative of H' in t being equal to zero at points of maximal intensity.

COROLLARY 1. *Under the assumptions of the previous theorem, there exists at most one viscosity solution of* (4) *verifying $u = \varphi$ on $\partial\Omega$.*

This result may be applied to our case as soon as we assume that the luminous intensity I is Lipschitz continuous, which gives the proper smoothness to the Hamiltonian, and lower than 1 in order to ensure the existence of the strict viscosity subsolution. As a matter of fact, when the luminous intensity is equal to 1, there does not exist p in $\mathbb{R}^2$ such that

$$\sqrt{1+|p|^2} - (\alpha, \beta)\cdot p - \gamma < 0.$$

On the contrary, in case I is lower than 1, $\underline{u}(x,y) = (\alpha/\gamma)x + (\beta/\gamma)y$ is a strict $\mathbf{C}^2$ subsolution of the equation.

This is an optimal result for we can prove that as soon as there exists a point of maximal intensity the solution of the equation is no longer unique. This is obvious in the case of a purely vertical lighting: if u is a nonidentically equal to zero differentiable solution (i.e., if I is equal to 1 at least at one point but not identically equal to 1), then $-u$ is another solution, as the Hamiltonian only depends on $|\nabla u|^2$.

This suggests the type of supplementary information which may be used in order to fully determine the solution: we consider the open subset $\Omega' \subset \Omega$ defined by

$$\Omega' = \{x \in \Omega / I(x) \neq 1\}.$$

The equation

$$\begin{cases} H(x, \nabla u(x)) = 0 & \text{in } \Omega', \\ u = \varphi & \text{on } \partial\Omega' \end{cases}$$

has at most one viscosity solution. Therefore, if we know the value of the solution on $\partial\Omega'$, the shape is totally determined.

2.3. The Eikonal equation and existence results. Otherwise we need a condition on φ in order to ensure existence. We present below the "compatibility condition," which implies the existence of a viscosity solution, and the condition necessary for the existence of a $\mathbf{C}^1$ solution; these were given in [16].

These results were established for the equations of the form $|\nabla u| = n$, where n is a continuous function on $\bar{\Omega}$, to which (1) can be easily reduced in case of purely vertical lighting.

For oblique lightings, one may notice that a simple change of references in $\mathbb{R}^3$ which rotates the axis in order that the illuminant direction and the z-axes coincide, reduces the equation to the vertical case. Yet, if I is not greater than $\sqrt{\alpha^2+\beta^2}$ on the entire set $\bar{\Omega}$, the shape may not be a function in those new references. In the following, we will consider that

$$I > \sqrt{\alpha^2+\beta^2},$$

which allows us to restrict the study to the vertical lighting.

Then, setting

$$n(x) = \sqrt{\frac{1}{I(x)^2} - 1} \quad \text{on } \bar{\Omega},$$

the equations (1), (2) taken for $\alpha = \beta = 0$ reduce to

$$\begin{cases} |\nabla u(x)| = n(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

We shall first consider the following simple situation: let Ω be a bounded, smooth, and connected domain and let $n \in \mathbf{C}(\bar{\Omega})$ be a positive function except at x_0 ($n(x_0) = 0$); we are looking for the viscosity solutions of (5).

Let L be the continuous function defined on $\bar{\Omega} \times \bar{\Omega}$ by

$$L(x, y) = \inf\left\{ \int_0^{T_0} n(\xi(s))\, ds / \xi(0) = x,\ \xi(T_0) = y, \right.$$
$$\left. \left|\frac{d\xi(t)}{dt}\right| \leqq 1,\ \xi(t) \in \bar{\Omega}, \forall t \in [0, T_0] \right\}.$$

PROPOSITION 1. *A viscosity solution of* (5) *satisfying the boundary condition* $u(x_0)=\lambda$, *where* λ *is a given real, may be written as the value function of a particular optimal control problem*

$$u(x)=\min\left\{\inf_{y\in\partial\Omega} L(x,y),\ \lambda+L(x,x_0)\right\}. \tag{6}$$

In addition, the following compatibility condition with boundary data holds:

$$|\lambda|\leqq \inf_{y\in\partial\Omega} L(x_0,y). \tag{7}$$

Conversely, if (7) *holds, the value function defined by* (6) *is a viscosity solution of* (5) *satisfying* $u(x_0)=\lambda$.

PROPOSITION 2. *Let* $u\in \mathbf{C}^1(\bar\Omega)$ *be a solution of* (5), *then we have for all* $x\in\bar\Omega$, *either*

$$u(x)=\inf_{y\in\partial\Omega} L(x,y)$$

or

$$u(x)=-\inf_{y\in\partial\Omega} L(x,y).$$

In particular, if $u\in\mathbf{C}^1(\bar\Omega)$ *is a solution of* (5) *then*

$$|u(x_0)|=\inf_{x\in\partial\Omega} L(x,x_0).$$

We conclude by mentioning that the previous results may be extended to the case where $n(x)>0$ in $\bar\Omega-\{x_1,\cdots,x_m\}$ and $n(x_i)=0$ for $i=1$ to m, where $x_1,\cdots,x_m$ are m distinct points of Ω:

(i) In this case, the compatibility condition is

$$|u(x_i)|\leqq \inf_{y\in\partial\Omega} L(y,x_i) \quad \forall i\in\{1\cdots m\}$$

and

$$|u(x_i)-u(x_j)|\leqq L(x_i,x_j) \quad \forall i,j\in\{1\cdots m\};$$

(ii) If $u\in\mathbf{C}^1(\bar\Omega)$ is a solution of (5) then $\exists i_0\in\{1\cdots m\}$ such that

$$|u(x_{i_0})|=\inf_{y\in\partial\Omega} L(y,x_{i_0})$$

and for all $i\in\{1\cdots m\}$ either

$$|u(x_i)|=\inf_{y\in\partial\Omega} L(y,x_i)$$

or

$$\exists j\in\{1\cdots m\},\quad j\neq i,\quad \text{s.t. } |u(x_i)-u(x_j)|=L(x_i,x_j)$$

holds.

Note that those results may be extended to the case where $n=0$ on closed subsets of $\bar\Omega$.

3. Numerical approximation. In this section we present a numerical scheme to compute the unique viscosity solution of

$$\begin{cases} |\nabla u(x)|=n(x) & \text{in } \Omega', \\ u(x)=\varphi & \text{on } \partial\Omega'. \end{cases} \tag{8}$$

The approach, based on the dynamic programming principle, yields monotone, stable, and consistent schemes whose convergence was originally proved in some situations by Crandall and Lions [11], Souganidis [21], and Barles and Souganidis [5].

As the scheme is implicit we then use a method, proposed by Osher and Rudin [17], to deduce an explicit scheme which has the great advantage of being fast.

3.1. Optimal control theory and dynamic programming principle. It is now well known, after the work of Lions [16], that in some situations a viscosity solution may be considered as the value function of an optimal control problem. We shall use here the formulation of u in term of value function and, more precisely, the dynamic programming principle of Bellman, in order to construct a monotone and consistent scheme.

Let u be the viscosity solution of (8). With the remark that

$$|\nabla u(x)| = n(x) \Leftrightarrow \sup_{|q|\leqq 1} \{\nabla u(x) \cdot q - n(x)\} = 0 \quad \forall x \in \Omega',$$

u appears to be the value function of the following exit time problem: let y_x be the state of the controlled dynamical system,

$$\begin{cases} \dot{y}_x = -q(s), & s \geqq 0, \\ y_x(0) = x \end{cases}$$

where q, the control, belongs to

$$\mathcal{A}_{adm} = \{q : \mathbb{R}_+ \to \mathbb{R}^2 \text{ measurable}/|q(s)| \leqq 1,\ s \geqq 0\}.$$

The cost function is defined by

$$J(x, q(\,)) = \int_0^T n(y_x(s))\, ds + \varphi(y_x(T)),$$

where T denotes the first exit time of Ω', i.e.,

$$T = \min\{t \geqq 0/y_x(t) \in \partial\Omega'\}.$$

Finally the value function is

$$u(x) = \inf_{q(\,)\in\mathcal{A}_{adm}} J(x, q(\,)).$$

Note that u is well defined because it always exists a control such that the trajectory y_x exits from Ω' in a finite time.

The dynamic programming principle allows us to assert that, for each $\tau \geqq 0$,

$$u(x) = \inf_{q(\,)\in\mathcal{A}_{adm}} \left\{ \int_0^{T\wedge\tau} n(y_x(s))\, ds + \varphi(y_x(T))\mathbb{1}_{\{T\leqq\tau\}}(T) + u(y_x(\tau))\mathbb{1}_{\{T>\tau\}}(T) \right\}, \tag{9}$$

where $\mathbb{1}$ is defined as follows:

Let A be a subset of $\mathbb{R}$; for all $t \in \mathbb{R}$, $\quad \mathbb{1}_A(t) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{if } t \notin A. \end{cases}$

3.2. Numerical scheme. We now present an explicit scheme which approximates the viscosity solution of (8) by discretizing (9) (see for more details Capuzzo-Dolcetta [6], Falcone [13], and Alziary de Roquefort [1]).

We shall use the formulation of u as a value function in order to approximate the nonlinear term.

Let Ω be the rectangular domain $]0, X[\times]0, Y[$ of $\mathbb{R}^2$. Given the mesh sizes Δx, $\Delta y > 0$, the value of our numerical approximation of the solution at $(x_i, y_j) = (i\Delta x, j\Delta y)$ ($i = 0 \cdots N$, $j = 0 \cdots M$, with $N = X/\Delta x$, $M = Y/\Delta y$) will be denoted by U_{ij}; N_{ij} will be the value of n at (x_i, y_j).

Finally we define the following index sets:

$$Q' = \{(i, j) \in \mathbb{N}^2 / (x_i, y_j) \in \Omega'\},$$

$$\partial Q' = \{(i, j) \in \mathbb{N}^2 / (x_i, y_j) \in \partial\Omega'\},$$

$$\bar{Q} = \{(i, j) \in \mathbb{N}^2 / (x_i, y_j) \in \bar{\Omega}\}.$$

We shall compute u in $\bar{\Omega}$ thanks to the dynamic programming principle choosing $\tau = \Delta t$.

If $x \in \partial\Omega'$, the first exit time is zero and we have

$$u(x) = \varphi(x).$$

If $x \in \Omega'$, we may choose Δt sufficiently small ($\Delta t < \min(\Delta x, \Delta y)$) for $y_x(s)$ to stay inside Ω' as $s \in [0, \Delta t]$ because the speed state is limited by 1.

Then we have for all lattice points in Ω'

$$u(x) = \inf_{q() \in \mathcal{A}_{adm}} \left\{ \int_0^{\Delta t} n(y_x(s))\, ds + u(y_x(\Delta t)) \right\},$$

i.e., making the usual approximations,

$$\sup_{|q| \leqq 1} \left\{ \frac{u(x - \Delta t q) - u(x)}{-\Delta t} - n(x) \right\} = 0.$$

With the choice of $\Delta t = \Delta x \Delta y / \sqrt{\Delta x^2 + \Delta y^2}$, a convex linear approximation for $u(x - \Delta t q)$ yields the following first-order finite difference scheme:

For all approximations U, we state: for all $(i, j) \in Q'$,

$$D_x^+ U_{ij} = \frac{U_{i+1j} - U_{ij}}{\Delta x},$$

$$D_y^+ U_{ij} = \frac{U_{ij+1} - U_{ij}}{\Delta y},$$

$$D_x^- U_{ij} = \frac{U_{ij} - U_{i-1j}}{\Delta x},$$

$$D_y^- U_{ij} = \frac{U_{ij} - U_{ij-1}}{\Delta y}.$$

Let $g = (g_{ij})_{(i,j) \in Q'}$ be a vector of functions from $\mathbb{R}^4$ to $\mathbb{R}$ defined by

$$\forall (i, j) \in Q', \quad \forall (a, b, c, d) \in \mathbb{R}^4,$$
$$g_{ij}(a, b, c, d) = \sqrt{\max(a^+, b^-)^2 + \max(c^+, d^-)^2} - N_{ij}.$$

Then, a numerical approximation U of (8) will satisfy

$$\text{(10)} \qquad \begin{cases} U_{ij} = \varphi(x_i, y_j) & \forall (i, j) \in \partial Q', \\ g_{ij}(D_x^- U_{ij}, D_x^+ U_{ij}, D_y^- U_{ij}, D_y^+ U_{ij}) = 0 & \forall (i, j) \in Q'. \end{cases}$$

Remark 3. The theoretical accuracy of this kind of monotone and consistent scheme is proportional to the square root of Δt, even if in practice we observe a first-order accuracy of order Δt.

Remark 4. In order that the convergence be monotonically increasing, the algorithm must start with a subsolution verifying the proper boundary conditions.

To prove the convergence of the solutions of the discrete equation (10) toward the viscosity solution of (8) as the mesh sizes go to zero, we shall use arguments contained in Barles and Souganidis [5] and, more precisely, the following result, which we shall adapt to our case.

Barles and Souganidis [5] consider any approximation schemes of the form

$$S(\rho, x, u^\rho(x), u^\rho) = 0 \quad \text{in } \bar{\Omega},$$

where $S: \mathbb{R}^+ \times \bar{\Omega} \times \mathbb{R} \times B(\bar{\Omega}) \to \mathbb{R}$, where $B(\bar{\Omega})$ is the space of bounded functions defined on $\bar{\Omega}$. The authors prove that as long as these schemes are monotone, stable, and consistent with equation (3), they converge to the solution of this equation, provided that the problem admits a comparison principle.

For this purpose, they define the functions $\underline{u}$ and $\bar{u}$ as follows:

$$\forall x \in \bar{\Omega}, \quad \underline{u}(x) = \liminf_{\substack{\rho \to 0 \\ y \to x}} u^\rho(y), \qquad \bar{u}(x) = \limsup_{\substack{\rho \to 0 \\ y \to x}} u^\rho(y).$$

It is then proved in [5] that $\underline{u}$ and $\bar{u}$ are, respectively, the lower semicontinuous viscosity supersolution and the upper semicontinuous viscosity subsolution of (3) on Ω and use the comparison result to conclude that $\underline{u} = \bar{u} = u$.

We shall here consider the simple case in which $n > 0$ on $\Omega' = \Omega - \{x_0\}$, where $x_0 \in \Omega$ and $n(x_0) = 0$. This proof can be extended to more general cases as Propositions 1 and 2.

A discretization ρ will represent a pair $(\Delta x, \Delta y)$ of space discretization steps such that x_0 be a point of the mesh, and $\rho \to 0$ will mean $\sqrt{\Delta x^2 + \Delta y^2} \to 0$.

In our case, u^ρ is a function defined on $\Delta_\rho = \{(x_i, y_j),\ i = 0 \cdots N, j = 0 \cdots M\} \cap \bar{\Omega}$ by $u^\rho(x_i, y_j) = U_{ij}$; therefore we define $\underline{u}$ and $\bar{u}$ as follows: for all x in $\bar{\Omega}$,

$$\underline{u}(x) = \lim_{\eta \to 0} \inf_{0 < \rho < \eta} \inf_{y \in B(x, \eta) \cap \Delta_\rho} u^\rho(y)$$

and

$$\bar{u}(x) = \lim_{\eta \to 0} \sup_{0 < \rho < \eta} \sup_{y \in B(x, \eta) \cap \Delta_\rho} u^\rho(y).$$

The scheme is monotone in the sense given in [5], i.e., if $U \geqq V$ then for all $(i, j) \in Q'$ and for all $t \in \mathbb{R}$ we have

$$g_{ij}\left(\frac{t - U_{i-1j}}{\Delta x}, \frac{U_{i+1j} - t}{\Delta x}, \frac{t - U_{ij-1}}{\Delta y}, \frac{U_{ij+1} - t}{\Delta y}\right)$$
$$\leqq g_{ij}\left(\frac{t - V_{i-1j}}{\Delta x}, \frac{V_{i+1j} - t}{\Delta x}, \frac{t - V_{ij-1}}{\Delta y}, \frac{V_{ij+1} - t}{\Delta y}\right).$$

The scheme is stable in the sense that for all $\rho > 0$, u^ρ exists and has a bound independent of ρ. The existence of u^ρ and of a superior bound independent of ρ is obvious considering the definition and the monotonicity of g. Moreover, u^ρ is bounded below, for u^ρ admits no local minimum in Ω'. Indeed, if U_{ij} (for $(i, j) \in Q'$) is lower than its neighbors, then

$$g_{ij}(D_x^- U_{ij}, D_x^+ U_{ij}, D_y^- U_{ij}, D_y^+ U_{ij}) = -N_{ij} < 0.$$

Then u^ρ has its minima on $\partial\Omega'$ and is bounded below by a constant independent of ρ, which is the minimum of φ on $\partial\Omega'$.

The scheme is consistent with the equation

$$H(x, \nabla u) = 0 \quad \text{in } \Omega'$$

as, for each $(i, j) \in Q'$,

$$g_{ij}(a, a, b, b) = \sqrt{a^2 + b^2} - n(x_i, y_j) \quad \text{for } a, b \in \mathbb{R}.$$

Then, we can prove as in [5] that $\underline{u}$ and $\bar{u}$ are, respectively, the super- and subsolution of $|\nabla u| = n$ in Ω' and that $\underline{u} \leqq \varphi \leqq \bar{u}$ on $\partial\Omega'$.

THEOREM 2. *With the above definitions,* $\underline{u} = \bar{u}$ *and thus the scheme defined by* (10) *converges toward the viscosity solution of* (8).

We shall need the following lemmas, whose proofs do not present any difficulty and will not be displayed here.

LEMMA 1. *We have the following propositions*:
(i) $\underline{u}$ *and* $\bar{u}$ *are continuous in* Ω',
(ii) $\underline{u}$ *is continuous on* $\bar{\Omega}$,
(iii) $\bar{u} = \varphi$ *on* $\partial\Omega'$.

LEMMA 2. *Let* $x_1 \in \partial\Omega'$; *if* x_1 *is a local minimum of* $\underline{u}$ *on* $\bar{\Omega}$ *then* $\underline{u}(x_1) = \varphi(x_1)$.

As $\bar{u}$ is continuous in Ω' and upper semicontinuous on $\bar{\Omega}$, its lower semicontinuous envelope $\bar{u}_*$ is continuous on $\bar{\Omega}$ and is a viscosity subsolution of (8) in Ω' as well as $\bar{u}$. Thus, as $\bar{u}_* \leqq \bar{u} = \varphi$ on $\partial\Omega'$, the comparison theorem implies that $\bar{u}_* \leqq u$ on $\bar{\Omega}$. Finally, it yields

$$\underline{u} \leqq \bar{u}_* \leqq \bar{u} \leqq u \quad \text{on } \bar{\Omega}.$$

We shall now prove that $\underline{u}$ is greater than u on $\bar{\Omega}$. We shall consider the following two cases:

$$\varphi(x_0) \geqq 0 \quad \text{and} \quad \varphi(x_0) < 0.$$

In the first case, as u^ρ has no minimum outside $\partial\Omega'$ and as $u^\rho = 0$ on $\partial\Omega$, $u^\rho \geqq 0$ on Δ_ρ and thus $\underline{u} \geqq 0$ on $\bar{\Omega}$. Then $\underline{u} = 0$ on $\partial\Omega$ and, as $\underline{u}$ is a viscosity supersolution and

$$0 \leqq \underline{u}(x_0) \leqq \varphi(x_0) \leqq \inf_{y \in \partial\Omega} L(x_0, y),$$

the compatibility condition holds and we have

$$\min\left(\inf_{y \in \partial\Omega} L(x, y), \underline{u}(x_0) + L(x, x_0)\right) \leqq \underline{u}(x) \quad \text{in } \bar{\Omega}$$

thanks to Proposition 1 and Theorem 1.

If x_0 is not a local minimum of $\underline{u}$, we have:

$$\begin{aligned}
&\forall \eta > 0, \quad \exists x \in B(x_0, \eta) / \underline{u}(x) < \underline{u}(x_0) \\
&\Rightarrow \forall \eta > 0, \quad \exists x \in B(x_0, \eta) \Big/ \inf_{y \in \partial\Omega} L(x, y) < \underline{u}(x_0) \\
&\Rightarrow \underline{u}(x_0) = \inf_{y \in \partial\Omega} L(x_0, y)
\end{aligned}$$

for $x \to \inf_{y \in \partial\Omega} L(x, y)$ is a continuous function. Now

$$\varphi(x_0) \leqq \inf_{y \in \partial\Omega} L(x_0, y) \quad \text{and} \quad \underline{u}(x_0) \leqq \varphi(x_0);$$

then $\underline{u}(x_0) = \varphi(x_0)$ and $\underline{u} \geqq u$ on $\bar{\Omega}$ thanks to Theorem 1.

If x_0 is a local minimum of $\underline{u}$ then Lemma 2 states that $\underline{u}(x_0) = \varphi(x_0)$ and we have the same conclusion.

Finally, in case $\varphi(x_0)<0$, u^ρ has a minimum at x_0 and $\underline{u}(x_0)=\varphi(x_0)$. Then the proof will be completed if $\underline{u}\geqq 0$ on $\partial\Omega$.

Let x_1 be the minimum of $\underline{u}$ on $\partial\Omega$; as $0\leqq \underline{u}(x_1)-\underline{u}(x_0)\leqq \inf_{y\in\partial\Omega} L(x_0,y)$, the compatibility condition is satisfied and we have

$$\min\,(\underline{u}(x_1)+\inf_{y\in\partial\Omega} L(x,y),\ \varphi(x_0)+L(x,x_0))\leqq \underline{u}(x)\quad \text{in } \bar{\Omega}.$$

Then again, either x_1 is a local minimum of $\underline{u}$ on $\bar{\Omega}$ and $\underline{u}(x_1)=0$ thanks to Lemma 2 or for all $\eta>0$, $\exists x\in B(x_1,\eta)/\underline{u}(x)<\underline{u}(x_1)$, i.e.,

$$\underline{u}(x_0)+L(x,x_0)<\underline{u}(x_1).$$

And, as L is continuous,

$$L(x_1,x_0)<\underline{u}(x_1)-\underline{u}(x_0)\leqq \inf_{y\in\partial\Omega} L(x_0,y)$$

$$\Rightarrow L(x_1,x_0)=\inf_{y\in\partial\Omega} L(x_0,y)=\underline{u}(x_1)-\underline{u}(x_0)$$

$$\Rightarrow 0\geqq \underline{u}(x_1)=\underline{u}(x_0)+\inf_{y\in\partial\Omega} L(x_0,y)\geqq 0$$

$$\Rightarrow \underline{u}(x_1)=0.$$

Thus, $\underline{u}\geqq u$ on $\partial\Omega$.

3.3. Numerical algorithm. We shall now present an algorithm which computes an approximation U solution of (10) for a given $(\Delta x,\Delta y)$. This algorithm, proposed by Osher and Rudin in [17], is efficient in the sense that it is explicit and that it requires very few iterations to converge.

Let G be the operator defined on the space of all $U=(U_{ij})_{(i,j)\in\bar{Q}}$ by: for all $(i,j)\in Q'$,

$$G(U)_{ij}=g_{ij}(D_x^- U_{ij},\ D_x^+ U_{ij},\ D_y^- U_{ij},\ D_y^+ U_{ij}).$$

U is an approximation of the viscosity solution of (8) for the discretization $(\Delta x,\Delta y)$ if $G(U)=0$ and $U_{ij}=\varphi(x_i,y_j)$ for $(i,j)\in\partial Q'$.

Let us first remark the following statements.

PROPOSITION 3. *For all* $(i,j)\in Q'$, *we have*:

(i) $U_{ij}\to G(U)_{ij}$ *is a continuous nondecreasing function*;

(ii) *For all* $(k,l)\neq(i,j)$, $U_{kl}\to G(U)_{ij}$ *is a continuous nonincreasing function*;

(iii) $\lim_{U_{ij}\to+\infty} G(U)_{ij}=+\infty$ *and for the choice* $U_{ij}=\min\,(U_{i+1j},U_{i-1j},U_{ij+1},U_{ij-1})$, *we have* $G(U)_{ij}=-N_{ij}<0$.

Therefore, if U is given at each point $(k,l)\neq(i,j)$, it is possible to find $t\in\mathbb{R}$ such that for $U_{ij}=t$, we have $G(U)_{ij}=0$.

Then, the algorithm will be the following:

(i) Step $n=0$: choose $U^0=(U_{ij}^0)_{(i,j)\in\bar{Q}}$ such that $U_{ij}^0=\varphi(x_i,y_j)$ for all $(i,j)\in\partial Q'$ and $G(U^0)\leqq 0$.

(ii) Step $n+1$: choose $(i,j)\in Q'$, set $\bar{V}=\sup\{V=(V_{kl})_{(k,l)\in\bar{Q}}$ for all $(k,l)\neq(i,j)$, $V_{kl}=U_{kl}^n$ and $G(V)_{ij}=0\}$. Set $U_{ij}^{n+1}=\bar{V}_{ij}$.

Remark 5. Note that step (ii) has to be completed an infinite number of times at each point of Q' to obtain the convergence toward the solution.

THEOREM 3. *This algorithm converges toward a solution of* (10) *for the discretization* $(\Delta x,\Delta y)$ *as* $n\to\infty$.

Indeed $(U^n)_{n\in\mathbb{N}}$ is nondecreasing and, as for all $n\in\mathbb{N}$, $G(U^n)\leqq 0$, Proposition 3 implies that $(U^n)_{n\in\mathbb{N}}$ is bounded and thus converges. Let U be its limit.

For all (i,j) in $\partial Q'$, $U(i,j)=\varphi(x_i,y_j)$. Furthermore, for all (i,j) in Q', $V\to G(V)_{ij}$ is continuous and, as $(U^n)_{n\in\mathbb{N}}$ converges, $(G(U^n)_{ij})_{n\in\mathbb{N}}$ converges as $n\to\infty$. Now, as each time the second step of the algorithm is completed in (i, j), $G(U^n)_{ij}=0$, then zero is the limit of a subsequence of $(G(U^n)_{ij})_{n\in\mathbb{N}}$. Thus $G(U)_{ij}=0$ and the proof is completed.

In our case, the value of $\bar{V}_{ij}$ can be found explicitly and exactly before any numerical computation. Indeed, it is easy to determine the roots (a^*, b^*, c^*, d^*) of the equation

$$\sqrt{\max(a^+, b^-)^2+\max(c^+, d^-)^2}=C,$$

where $(a, b, c, d)\in\mathbb{R}^4$ and C is a positive constant. Then, setting

$$a^*=\frac{V_{ij}-U^n_{i-1j}}{\Delta x}$$

$$b^*=\frac{U^n_{i+1j}-V_{ij}}{\Delta x}$$

$$c^*=\frac{V_{ij}-U^n_{ij-1}}{\Delta y}$$

$$d^*=\frac{U^n_{ij+1}-V_{ij}}{\Delta y},$$

we compute easily the largest V_{ij} verifying the above equalities for all the possible roots.

The simplest way to find a solution of (10) would have been to look for the fixed points of

$$U\to U-\Delta t G(U)$$

by using the following monotone, consistent, and explicit scheme:

$$U^{n+1}_{ij}=U^n_{ij}-\Delta t G(U)_{ij}, \quad \forall(i,j)\in Q', \quad \forall n\in\mathbb{N}, \tag{11}$$

but that method would have been much slower than our algorithm.

Choosing U^0 as in (i), this algorithm is also monotonically increasing. Capuzzo-Dolcetta and Falcone (see [12], [13], and [7]) proposed a way to speed up this kind of scheme that consists in choosing, at step $n+1$, the largest approximation of the form $\lambda U^{n+1}+(1-\lambda)U^n$ such that $G(U)\leqq 0$, where U^{n+1} is yielded by (11) and with $\lambda\geqq 1$.

This increases the convergence speed of the scheme (11), but is not very well adapted to our case for which it reduces only by half the number of iterations, since λ remains close to 1 in order that $G(U)_{ij}$ may be lower than zero around the points at which $n=0$.

Roughly speaking, the method proposed in [17] is twenty times faster than (11) and corresponds, in the above algorithm, to the choice of a λ which depends on the current point (i, j), and thus does not present the previous disadvantage due to the nonuniform speed on Q' observed in (11).

4. Results and perspectives.

4.1. Numerical results. We tested first our algorithm on basic uniformly continuous shapes (parabola, sinusoid, pyramid, and other simple analytic three-dimensional shapes) with both high and low gradients illuminated by purely vertical lighting.

For each example we give the grid size; the $\mathbf{L}^1$-error ε, the number of iterations required for the convergence n, the luminous intensity I on $\bar{\Omega}=[0,1]\times[0,1]$, and the boundary condition on $\partial\Omega'\backslash\partial\Omega$.

The reader may notice a first-order accuracy in the $\mathbf{L}^1$ norm and the efficiency of the method as far as the number of iterations is concerned compared to the usual algorithms like the one defined by (11).

Let us remark that the greatest loss of accuracy is observed near the boundary of Ω'. For instance, the reconstructed positive parabola (Fig. 1) presents a jump at its top where the elevation has been imposed, whereas in the negative one (Fig. 2), the same phenomenon may be noticed at the corners of the square Ω. The size of the jumps, which are concentrated on a single mesh around the boundary, tends to decrease as the discretization goes to zero (see Fig. 7). A specific post-treatment may be applied in order to suppress those peaks.

This algorithm may be improved by using a second-order essentially nonoscillatory correction as in [17]: tests performed on one-dimensional images yielded faster convergence and more accurate results; besides, this correction is particularly efficient at getting rid of the jumps described above.

Moreover, a continuous and nondifferentiable shape corresponding to a Lipschitz continuous intensity may also be recovered with a first-order accuracy. Figure 3 displays

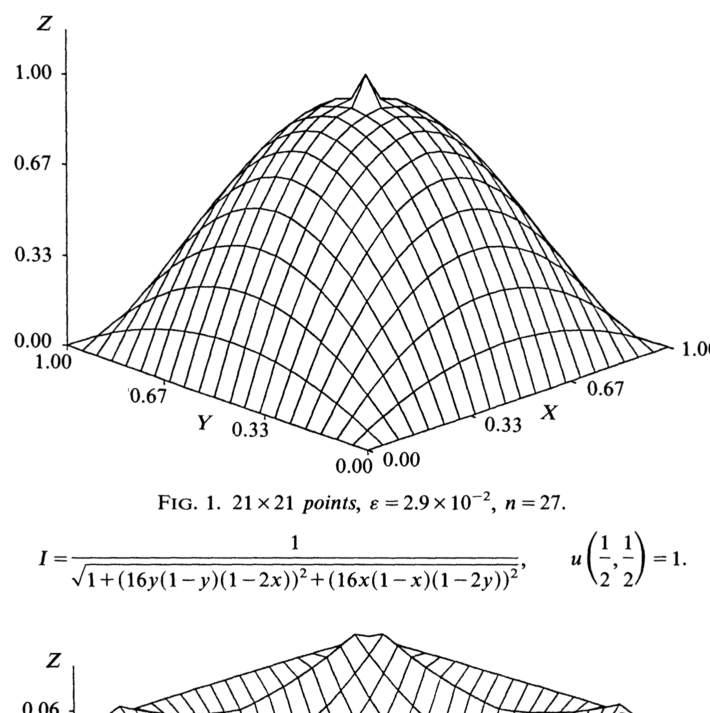

FIG. 1. 21×21 *points*, $\varepsilon = 2.9\times 10^{-2}$, $n=27$.

$$I=\frac{1}{\sqrt{1+(16y(1-y)(1-2x))^2+(16x(1-x)(1-2y))^2}},\qquad u\left(\frac{1}{2},\frac{1}{2}\right)=1.$$

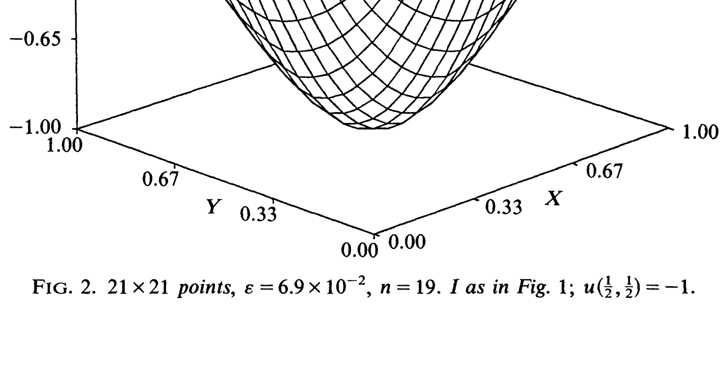

FIG. 2. 21×21 *points*, $\varepsilon = 6.9\times 10^{-2}$, $n=19$. *I as in Fig.* 1; $u(\frac{1}{2},\frac{1}{2})=-1$.

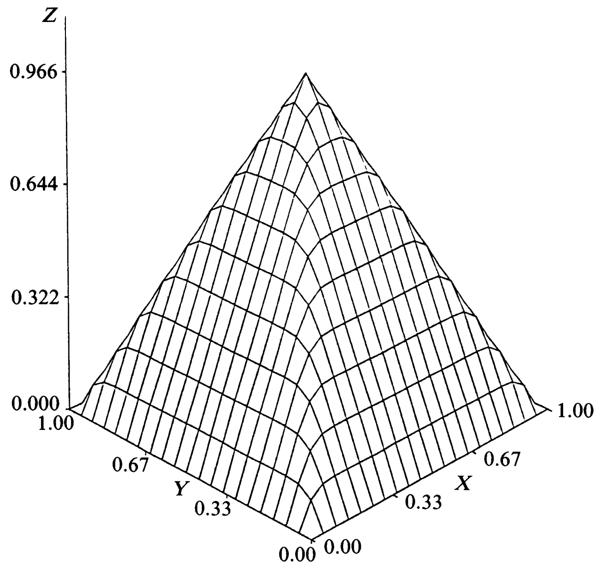

FIG. 3. 21×21 *points*, $\varepsilon = 1.1 \times 10^{-2}$, $n = 10$. $I = 1/\sqrt{5}$.

a regular pyramid that corresponds to the data of a constant intensity on the whole space and no additional information. Figure 4 represents the shape reconstructed from the intensity of a parabola with a given intermediate elevation at the point where $I = 1$.

We tested the robustness of the algorithm on the parabola and the pyramid by perturbing the luminous intensity after having detected the points where the elevation has to be given. We added a uniform noise on $[-\frac{1}{10}, \frac{1}{10}]$ to I and then truncated the obtained value so that the luminous intensity stays in $]0, 1]$.

Results are given in Figs. 5 and 6. The number of iterations and the smoothness of the shapes are affected, whereas the accuracy remains of order Δx.

Finally, when the information about the elevation at the singular points are not available, the results of § 2.3 may be used in order to recover directly all the $\mathbf{C}^1$ shapes.

Let us begin with the simple case when $\{I = 1\}$ reduces to a single point x_0. Then, as we saw in § 2.3, there exist at most two $\mathbf{C}^1$ solutions which are in fact the maximum and the minimum solutions. We can compute easily the shape which present a maximum by choosing "large near x_0" initial U^0. This gives thes the value of $\inf_{y \in \partial\Omega} L(x_0, y)$.

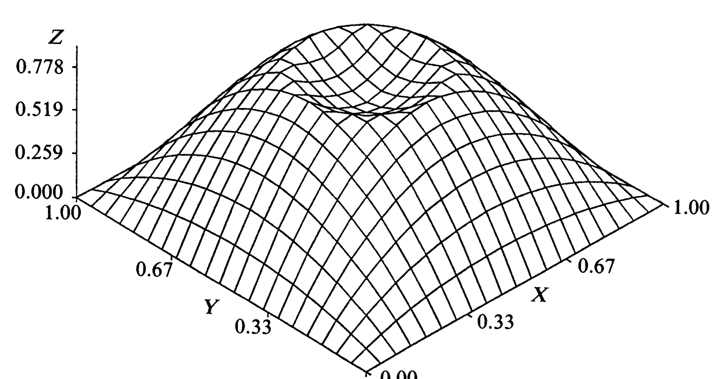

FIG. 4. 21×21 *points*, $\varepsilon = 4.6 \times 10^{-2}$, $n = 14$. *I as in Fig.* 1; $u(\frac{1}{2}, \frac{1}{2}) = \frac{47}{81}$.

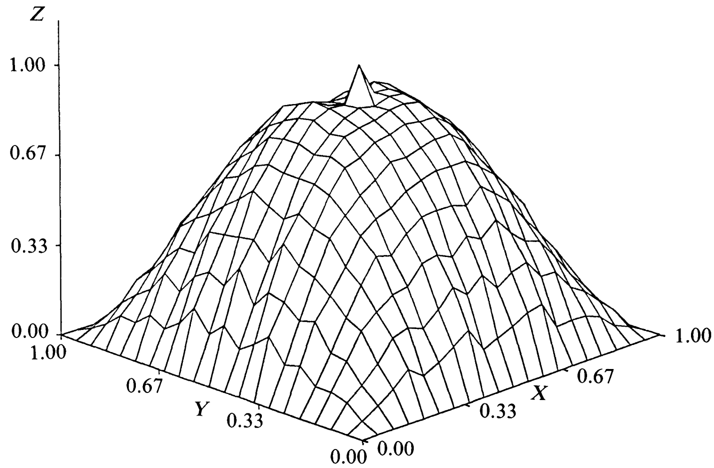

FIG 5. *Uniform noise on* I: 10%, $\varepsilon = 3.1 \times 10^{-2}$, $n = 43$. I *and* $u(\frac{1}{2}, \frac{1}{2})$ *as in Fig.* 1.

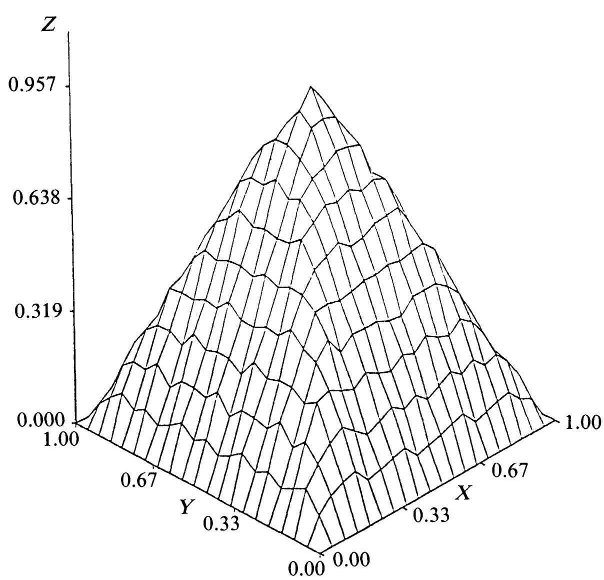

FIG. 6. *Uniform noise on* I: 10%, $\varepsilon = 1.2 \times 10^{-2}$, $n = 14$. I *as in Fig.* 3.

Then, we compute the other regular shape by imposing the Dirichlet boundary condition $u(x_0) = -\inf_{y \in \partial\Omega} L(x_0, y)$.

When $\{I = 1\}$ consists of several points, more combinations are possible (more $\mathbf{C}^1$ solutions can exist) and, as seen in § 2.3, they correspond to "extremal" elevations at those points. We compute first the maximal solution and deduce from the result the values of $L(x_i, x_j)$ and $\inf_{y \in \partial\Omega} L(x_i, y)$ for all (i, j). Then, we construct the other shapes, taking all the combinations of Dirichlet boundary conditions and selecting the smooth results. Figures 7 and 8 present two $\mathbf{C}^1$ shapes obtained with this method. Actually there exist four regular surfaces corresponding to the same intensity, the two others being the opposite shapes of the ones displayed here.

Actually, we did not try to reconstruct any shape thanks to the change of references proposed in § 2.3. Using the method displayed in § 3, we can build an algorithm that

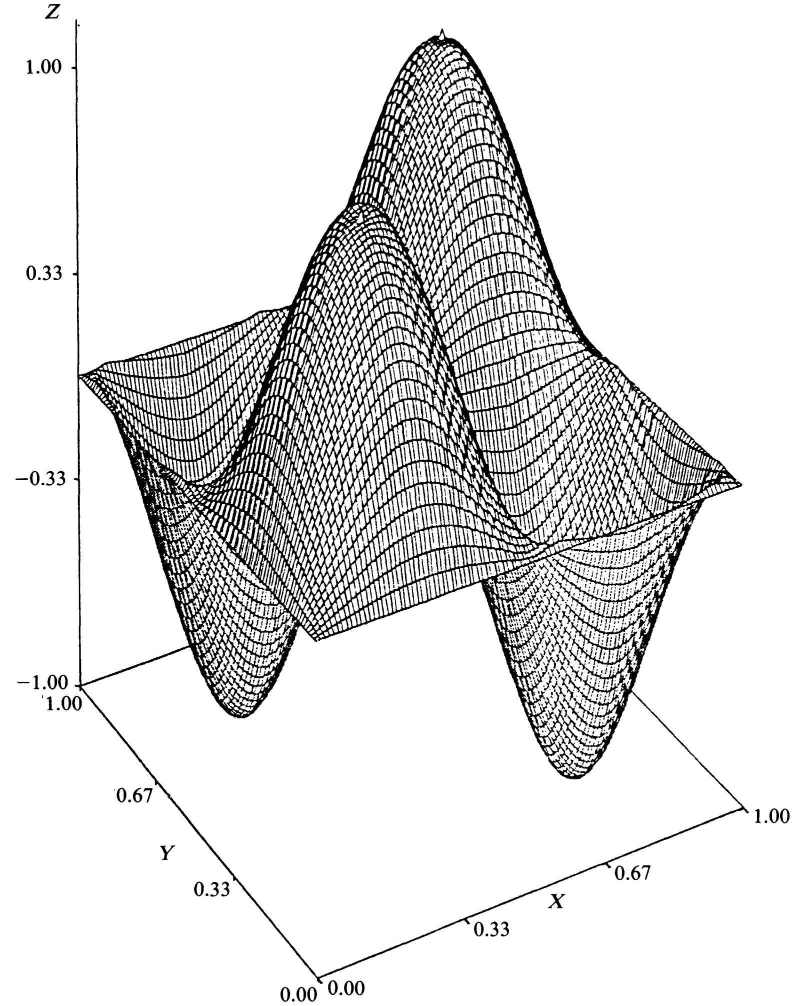

FIG 7. 101×101 *points*, $\varepsilon=2.6\times 10^{-2}$, $n=255$.

$$I=\frac{1}{\sqrt{1+(2\pi \sin(2\pi y)\cos(2\pi x))^2+(2\pi\sin(2\pi x)\cos(2\pi y))^2}},$$

$$u(\tfrac14,\tfrac14)=u(\tfrac34,\tfrac34)=1,\quad u(\tfrac14,\tfrac34)=u(\tfrac34,\tfrac14)=-1,\quad u(\tfrac12,\tfrac12)=0.$$

approximates (1), (2) in oblique cases with no restriction on I (see [17]). This has been done in one dimension and the results are as satisfactory as in purely vertical lighting. Figure 9 is an example in two dimensions obtained by a slightly different scheme which requires less preliminary computation but which is far slower.

4.2. Conclusion. We have presented here the simplest case in order to make the results as clear as possible. Without changing anything, we can adapt our results to close cases.

For example, as mentioned above, the same results may be used for recovering a surface illuminated from one side in such a way that I remains greater than $\sqrt{\alpha^2+\beta^2}$. The same remark can be done in case the illumination is no more punctual but considered as a uniform distribution of distant sources on a semisphere S. Indeed, the

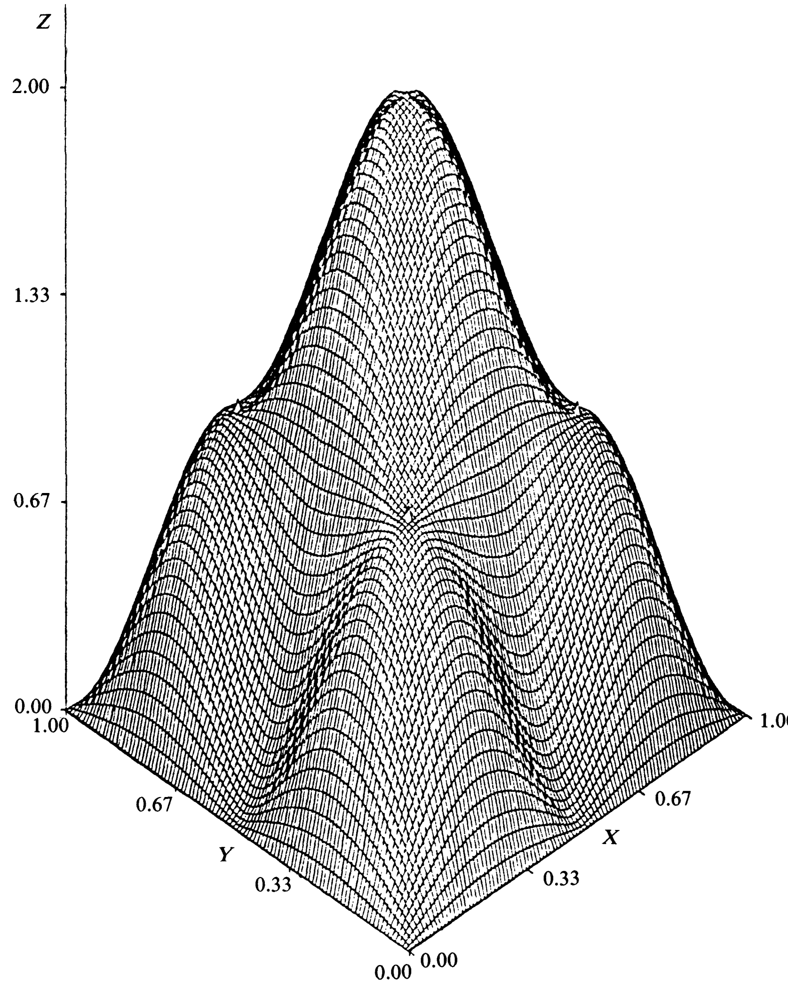

FIG. 8. 101×101 *points*, $n = 256$. *I as in Fig.* 7, *maximal solution*; $u(\frac{1}{4}, \frac{1}{4}) = u(\frac{3}{4}, \frac{3}{4}) = u(\frac{1}{4}, \frac{3}{4}) = u(\frac{3}{4}, \frac{1}{4}) = 1$, $u(\frac{1}{2}, \frac{1}{2}) = 2$.

image irradiance equation is written

$$I(x, y) = \frac{\int_S ((\partial u/\partial x) \cdot \omega_1 + (\partial u/\partial y) \cdot \omega_2 + \omega_3)\, d\mu(\omega)}{\sqrt{1+|\nabla u|^2}}.$$

And thus we have

$$I(x, y) = \frac{\bar{\omega}_1 \cdot (\partial u/\partial x) + \bar{\omega}_2 \cdot (\partial u/\partial y) + \bar{\omega}_3}{\sqrt{1+|\nabla u|^2}},$$

where

$$\bar{\omega}_i = \int_S \omega_i \, d\mu(\omega), \qquad i = 1, 2, 3,$$

which appears to be the same case as before if we renormalize ω.

In case the distribution of light sources is not uniform, it seems that the maximum principle still holds and therefore we can prove a uniqueness result as long as the support of the luminous source is connected.

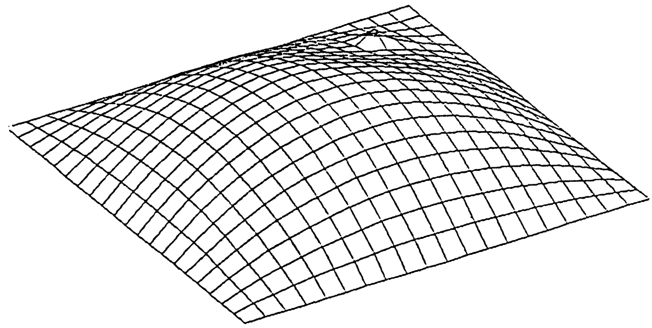

FIG. 9. 21×21 *points*, $\varepsilon = 5.9 \times 10^{-2}$, $n = 145$.

$$\alpha = \beta = \frac{2}{\sqrt{35}}, \qquad \gamma = \frac{3\sqrt{3}}{\sqrt{35}},$$

$$I = \frac{8y(1-y)(1-2x)+8x(1-x)(1-2y)+3\sqrt{3}}{\sqrt{35}\sqrt{1+(4y(1-y)(1-2x))^2+(4x(1-x)(1-2y))^2}},$$

$$u\left(\frac{1}{5}\right) = \frac{64}{625}.$$

We intended to study in further publications some variants of the model described above: in particular, Neumann type boundary conditions instead of Dirichlet ones, shapes presenting shadow areas, discontinuities of I, and light sources at a finite distance.

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